Multi-Agent Systems

Kai Cai

cai@omu.ac.jp

Consensus/
Rendezvous

Multi-agent system

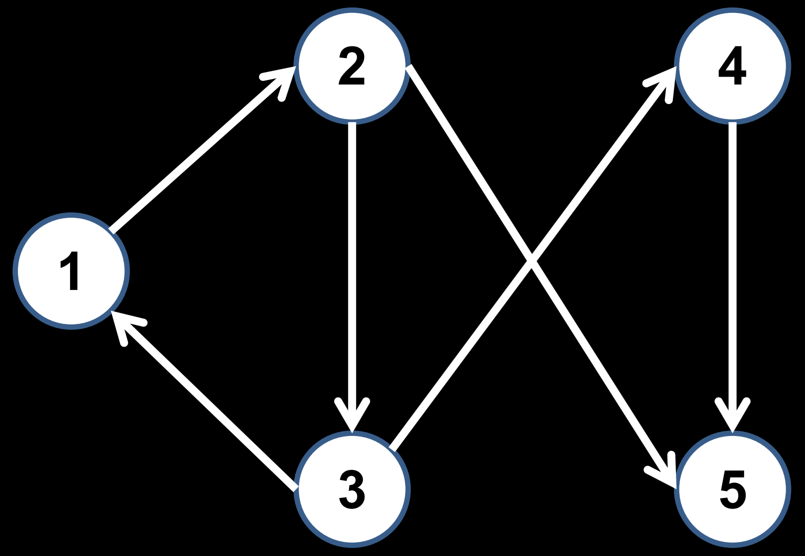
a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

node $v_i \in \mathcal{V}$: an agent

edge $(v_j, v_i) \in \mathcal{E}$: agent j sends information to v_i

example:



Consensus problem

each agent v_i updates its state based on

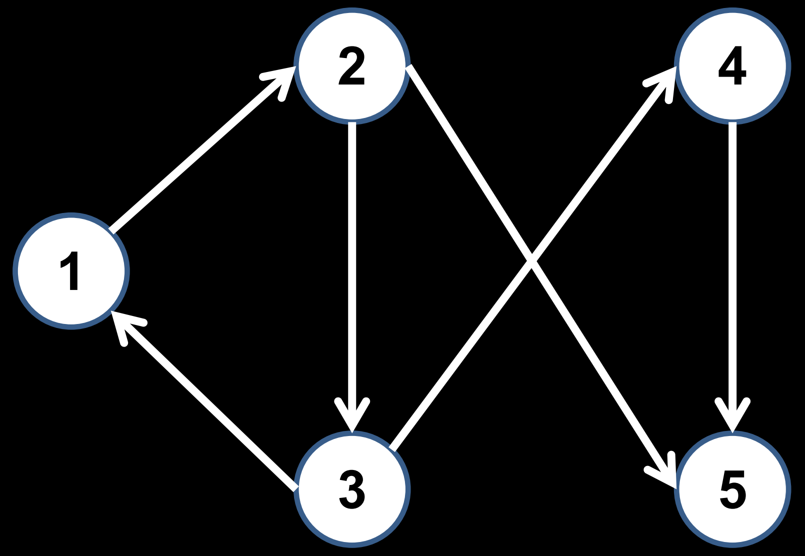
$$\dot{x}_i = u_i, \; x_i, u_i \in \mathbb{R}$$

consensus: design input $u_i(t)$, $t \geq 0$

s.t. $(\forall x_1(0), \ldots, x_n(0))(\exists c \in \mathbb{R}) x_i(t) \to c$

as $t \to \infty$

example:



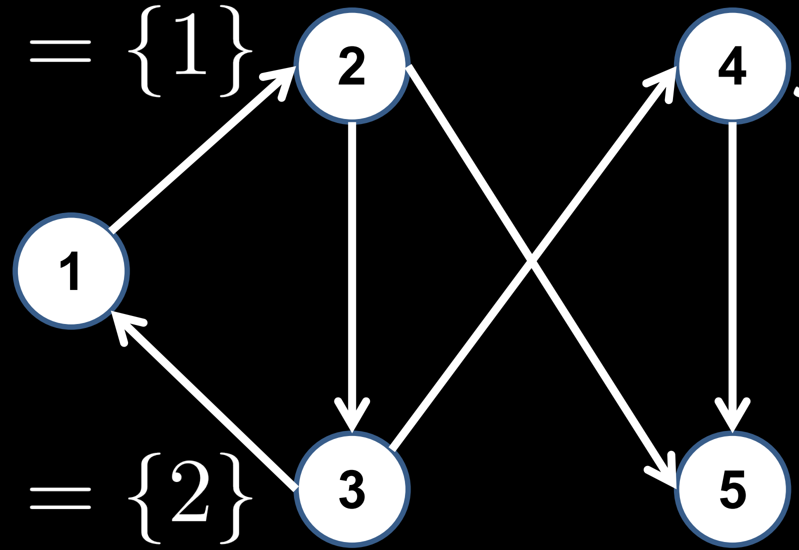
Distributed algorithm

each agent v_i can receive information $x_j(t)$ from neighbor(s) $j \in \mathcal{N}_i$

distributed algorithm: at time $t(\geq 0)$ design $u_i(t)$ based on information $x_j(t)$ where $j \in \mathcal{N}_i$

example: $\mathcal{N}_2 = \{1\}$ $\mathcal{N}_4 = \{3\}$

$\mathcal{N}_1 = \{3\}$



$\mathcal{N}_3 = \{2\}$ $\mathcal{N}_5 = \{2, 4\}$

Example

$$\dot{x}_1 = u_1 = x_3 - x_1$$
$$\dot{x}_2 = u_2 = x_1 - x_2$$
$$\dot{x}_3 = u_3 = x_2 - x_3$$
$$\dot{x}_4 = u_4 = x_3 - x_4$$
$$\dot{x}_5 = u_5 = (x_2 - x_5) + (x_4 - x_5)$$

example: $\mathcal{N}_2 = \{1\}$ $\mathcal{N}_4 = \{3\}$

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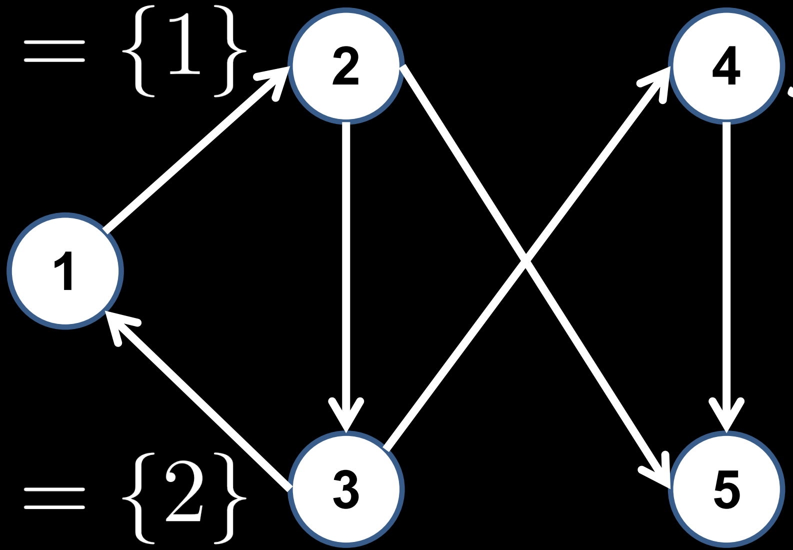

$\mathcal{N}_3 = \{2\}$ $\mathcal{N}_5 = \{2, 4\}$

Example

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$$\dot{x}_2 = u_2 = x_1 - x_2$$

$$\dot{x}_3 = u_3 = x_2 - x_3$$

$$\dot{x}_4 = u_4 = x_3 - x_4$$

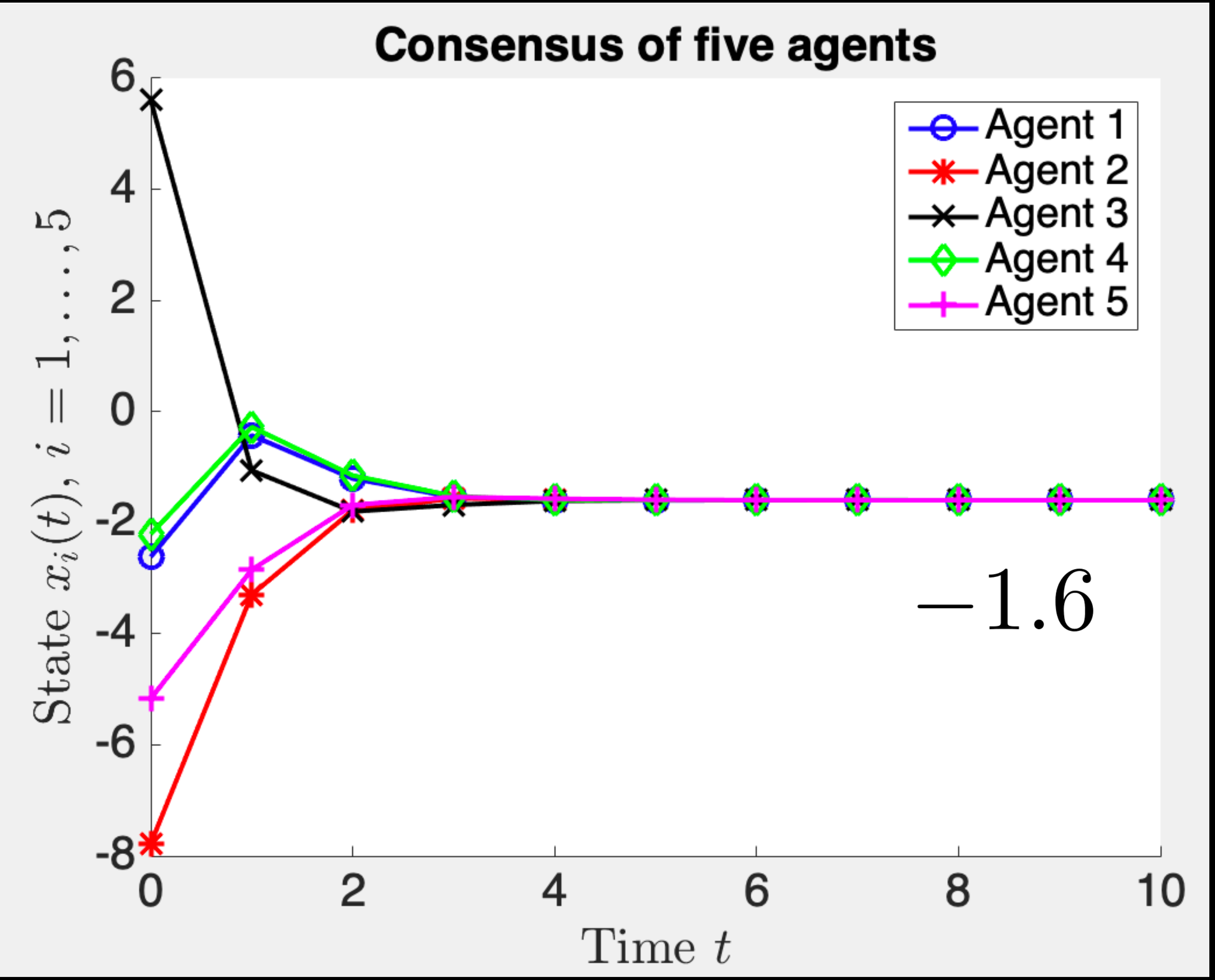
$$\dot{x}_5 = u_5 = (x_2 - x_5) + (x_4 - x_5)$$

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

relative state information

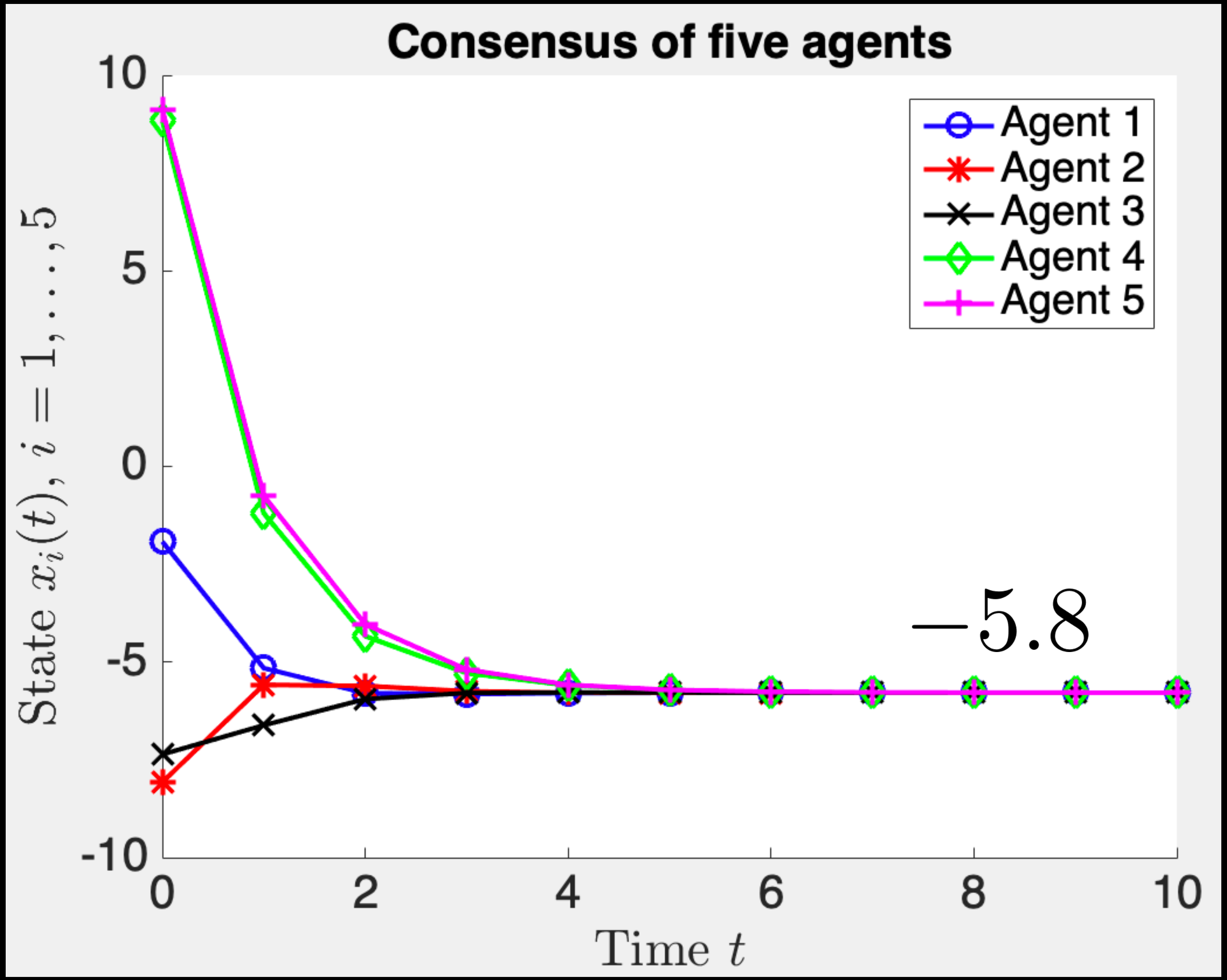
Example

$$\text{simulation: } x_1(0) = -2.6, x_2(0) = -7.8$$
$$x_3(0) = 5.6, x_4(0) = 2.2, x_5(0) = -5.2$$



Example

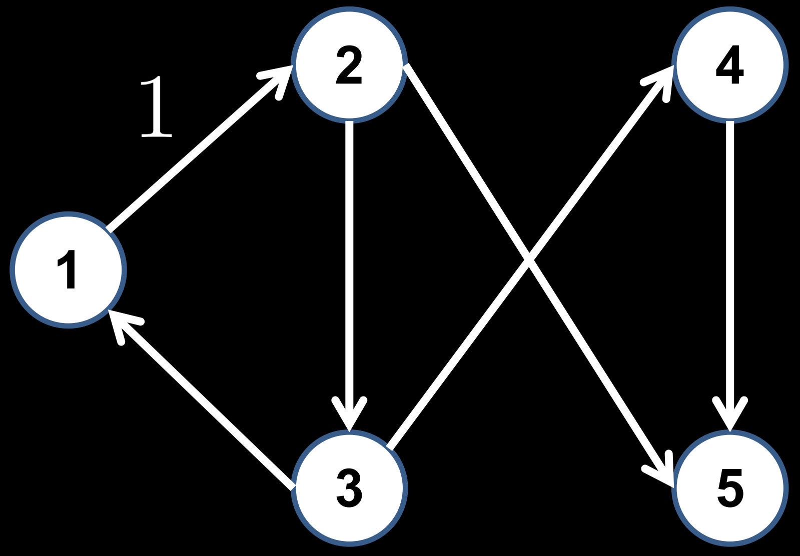
$$\text{simulation: } x_1(0) = -1.9, x_2(0) = -8.1$$
$$x_3(0) = -7.4, x_4(0) = 8.8, x_5(0) = 9.1$$



Weighted graph

example:

weighted graph $\mathcal{G}$



adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Weighted graph

example:

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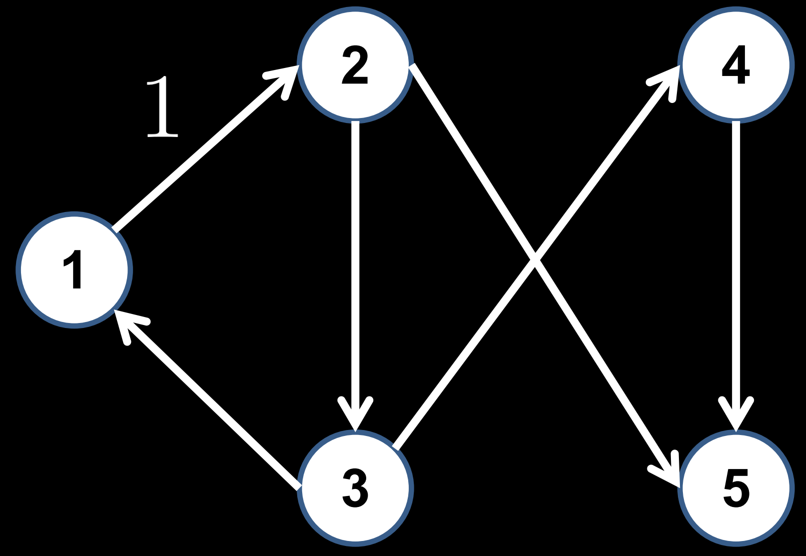

Laplacian matrix

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Equation

$$\dot{x}_1 = u_1 = x_3 - x_1$$
$$\dot{x}_2 = u_2 = x_1 - x_2$$
$$\dot{x}_3 = u_3 = x_2 - x_3$$
$$\dot{x}_4 = u_4 = x_3 - x_4$$
$$\dot{x}_5 = u_5 = (x_2 - x_5) + (x_4 - x_5)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix}}_{-L} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Recap, generalization

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}$$

Problem: design u_i to update x_i

s.t. $(\forall v_i \in \mathcal{V})(\forall x_i(0))(\exists c) \lim_{t \to \infty} x_i(t) = c$

Recap, generalization

a system of n interacting agents

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each agent v_i updates its state based on

$$\dot{x}_i = u_i,\ x_i, u_i \in \mathbb{R}$$

Distributed algorithm

$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$

based on information $x_j(t)$ or

relative information $x_j(t) - x_i(t)$

from neighbor agent(s) $j \in \mathcal{N}_i$

Recap, generalization

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \; x_i, u_i \in \mathbb{R}$$

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{x} = -Lx$$

Theorem

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i,\ x_i, u_i \in \mathbb{R}$$

$\dot{x} = -Lx$ solves consensus

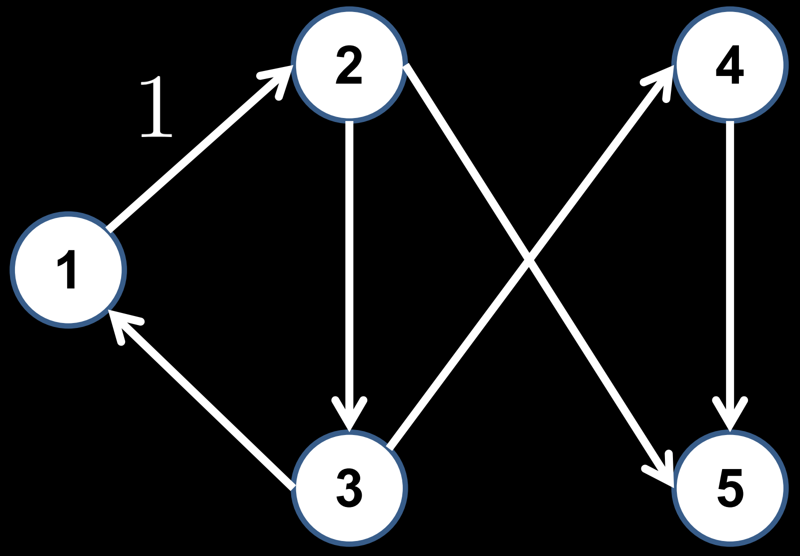
s.t. $(\forall v_i \in \mathcal{V})(\forall x_i(0))(\exists c) \lim_{t \to \infty} x_i(t) = c$

iff $\mathcal{G}$ contains a spanning tree

Example

example:

weighted graph $\mathcal{G}$



spanning tree (?)

Example

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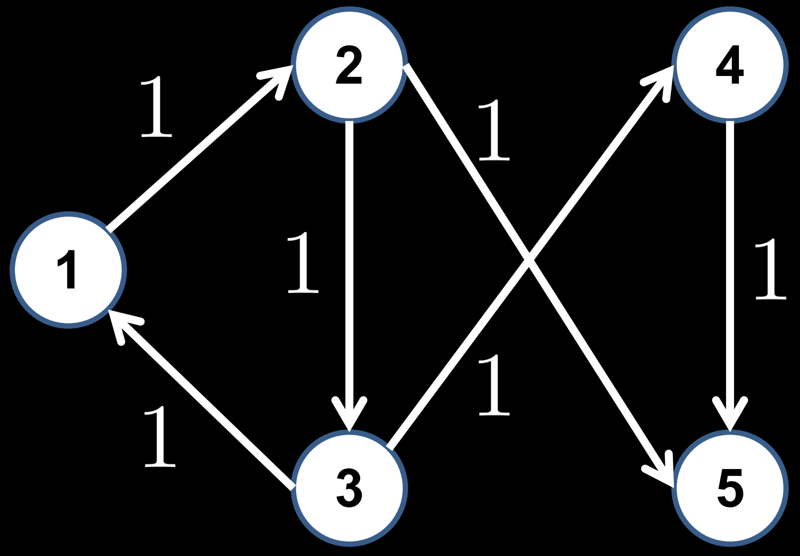
weighted graph $\mathcal{G}$



spanning tree (?)

Example

ordinary Laplacian matrix

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

$$\text{spanning tree} \Rightarrow \text{rank}(L) = n - 1$$

Example

ordinary Laplacian matrix

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

spanning tree $\Rightarrow \text{rank}(L) = 4$

zero row sum, i.e. $L\underline{\mathbf{1}} = 0$

eigenvector of eigenvalue 0

Example

ordinary Laplacian matrix

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

spanning tree $\Rightarrow \text{rank}(L) = 4$

nonzero column sum, i.e. $\mathbf{1}^\top L \neq 0$

rather $\underline{[\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ 0 \ \ 0]}L = 0$

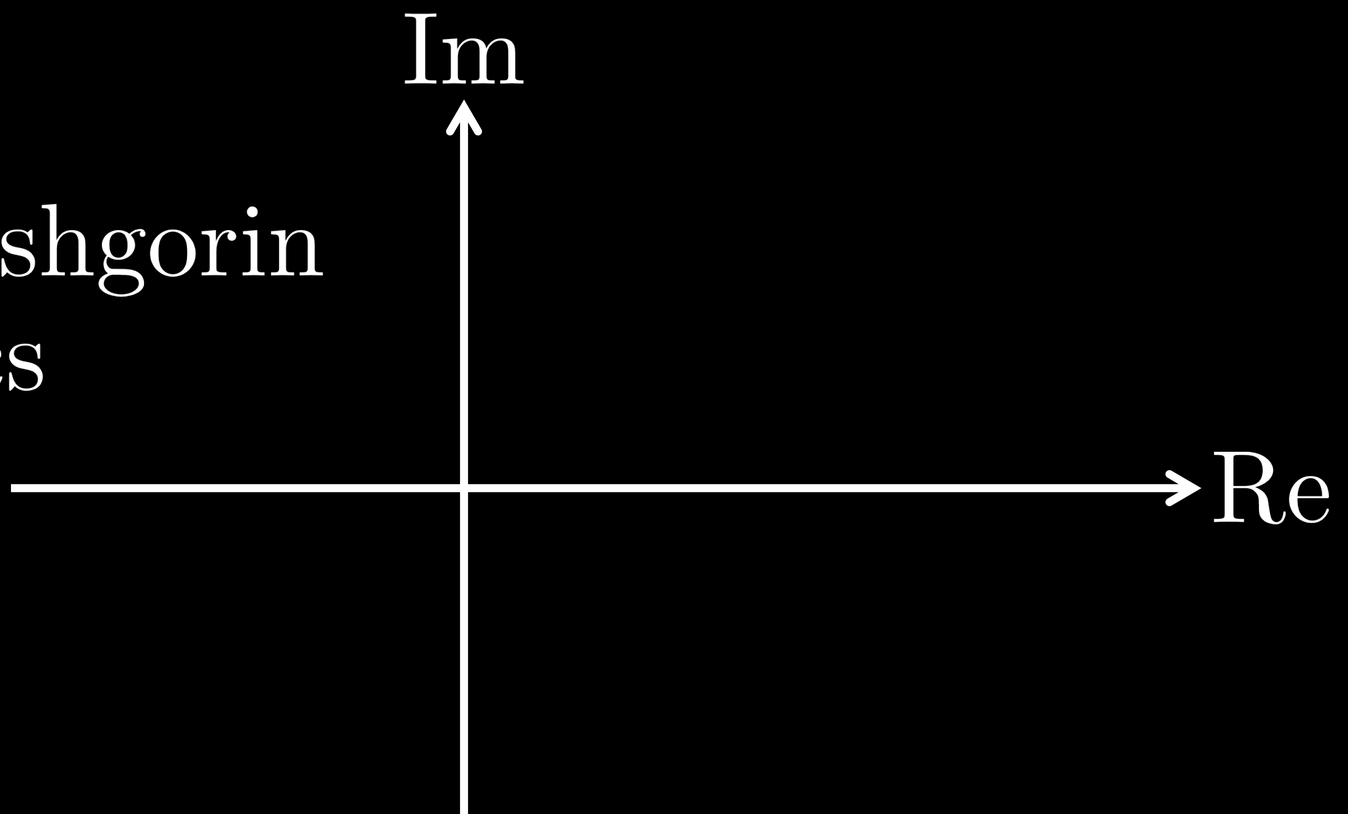
left eigenvector of eigenvalue 0

Example

eigenvalues

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Gershgorin discs

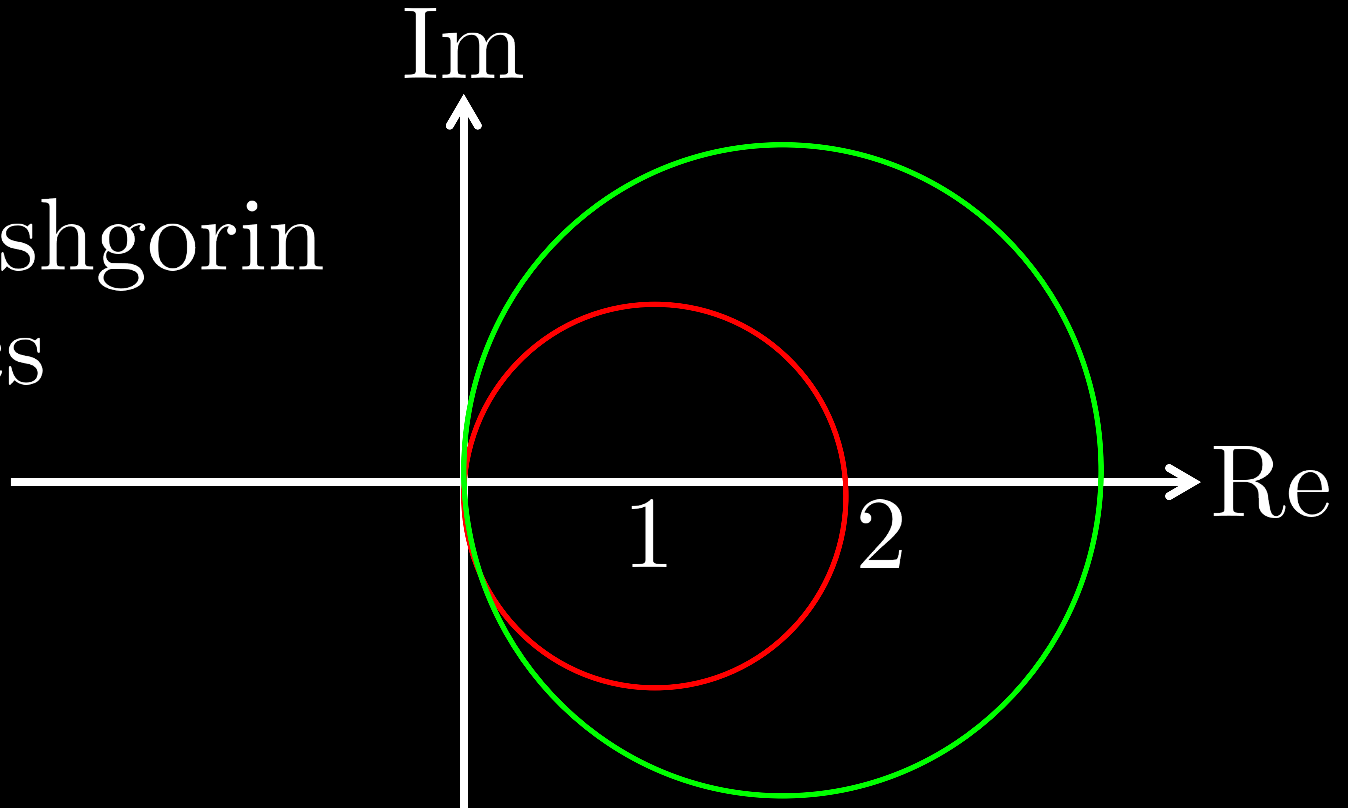


Example

eigenvalues

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Gershgorin discs

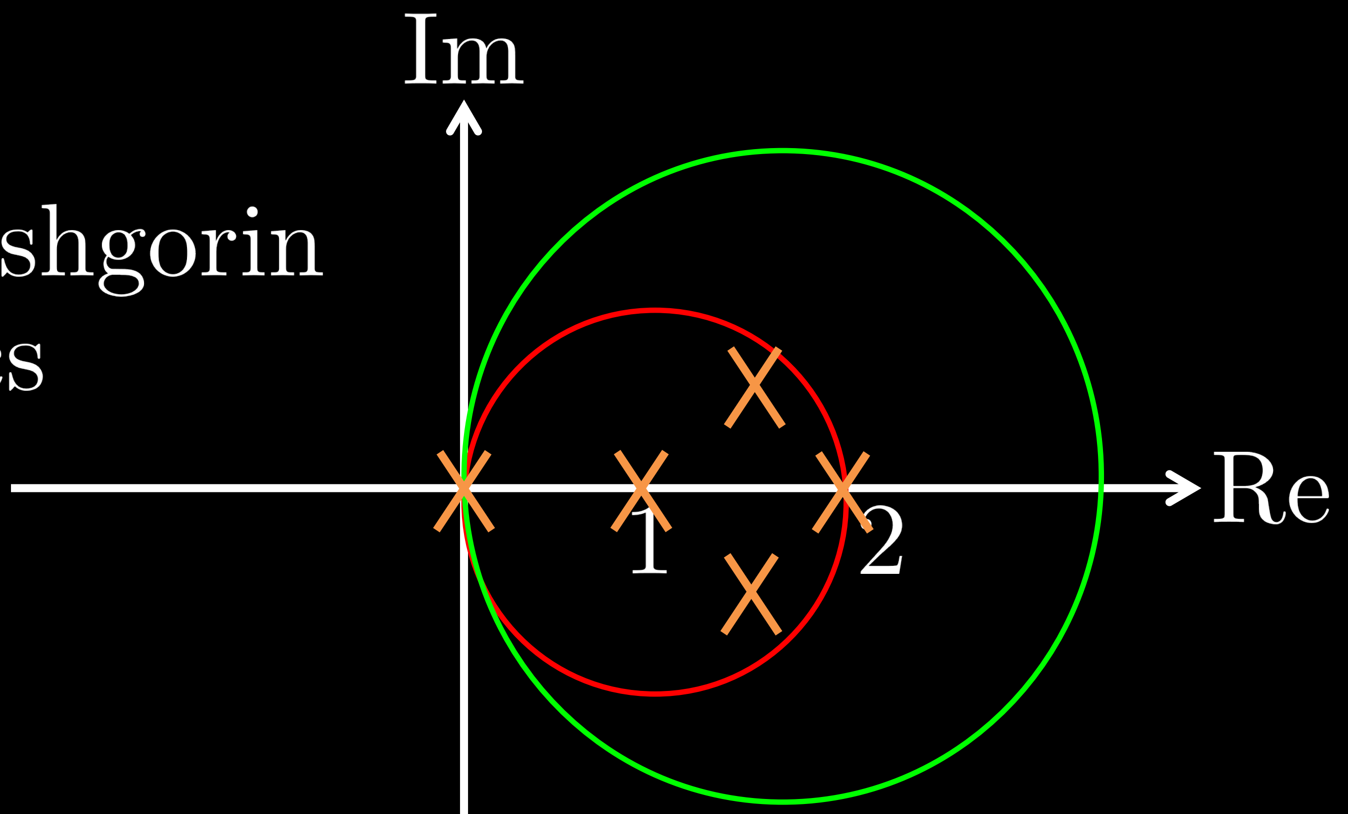


Example

eigenvalues of L

$$0, 1, 2, \tfrac{3}{2} \pm \tfrac{\sqrt{3}}{2}\mathrm{j}$$

Gershgorin discs

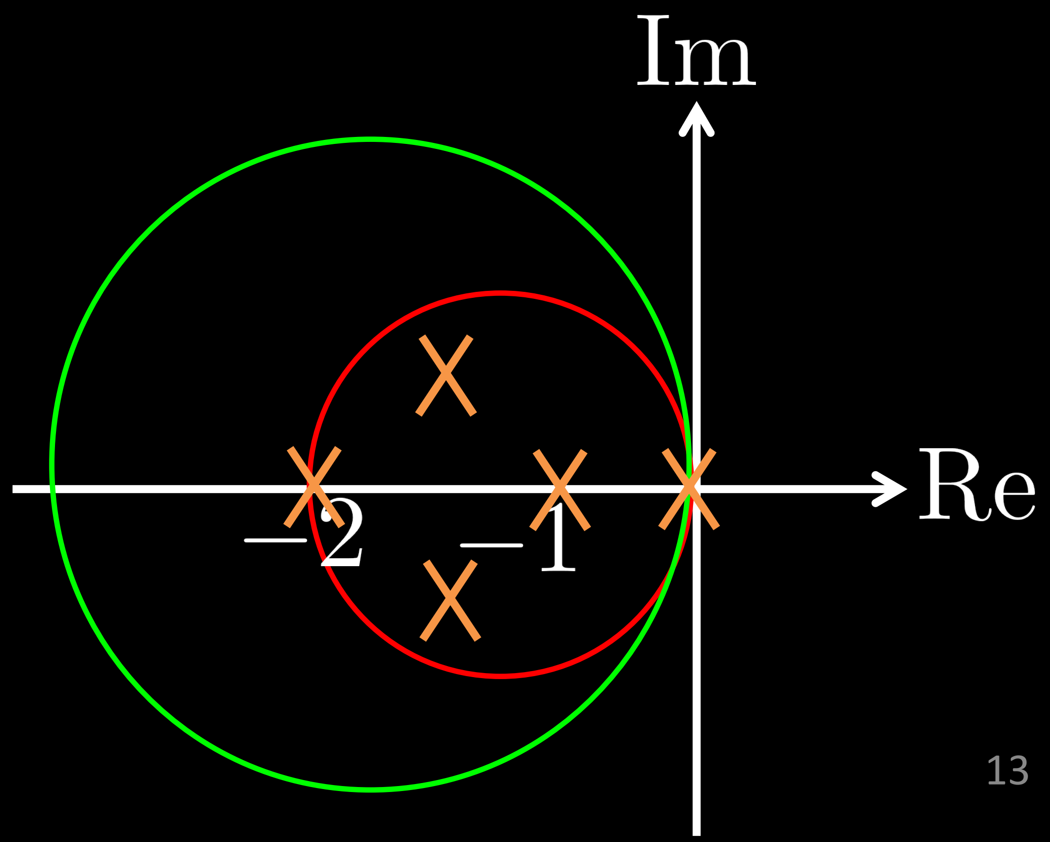


Example

eigenvalues of $-L$ (spectrum mapping)

$$0, -1, -2, -\frac{3}{2} \pm \frac{\sqrt{3}}{2}\mathrm{j}$$

Gershgorin discs



Example

diagonalization

$$-L = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$$= VJV^{-1}$$

Example

diagonalization

$$-L = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$$= VJV^{-1}$$

$$= [v_1 \;\; v_2 \;\; v_3 \;\; v_4 \;\; v_5] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} + \frac{\sqrt{3}}{2}\mathrm{j} & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} - \frac{\sqrt{3}}{2}\mathrm{j} \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix}$$

$$(w_1^\top v_1 = 1,\; w_1^\top v_2 = \cdots = w_1^\top v_5 = 0)$$

$$v_1 = \mathbf{1},\; w_1 = [\tfrac{1}{3} \;\; \tfrac{1}{3} \;\; \tfrac{1}{3} \;\; 0 \;\; 0]^\top$$

Example

$$\dot{x} = -Lx,\; x(0)$$

$$x(t) = ?$$

Example

$$\dot{x} = -Lx,\ x(0)$$

$$x(t) = \mathrm{e}^{-Lt}x(0)$$

matrix exponential:

$$\mathrm{e}^{-Lt} = I + (-Lt) + \tfrac{1}{2!}(-Lt)^2 + \tfrac{1}{3!}(-Lt)^3 + \cdots$$

Example

$$\dot{x} = -Lx, \; x(0)$$

$$x(t) = \mathrm{e}^{-Lt} x(0)$$

$$= \mathrm{e}^{VJV^{-1}t} x(0)$$

matrix exponential:

$$\mathrm{e}^{VJV^{-1}t} = I + (VJV^{-1}t) + \frac{1}{2!}(VJV^{-1}t)^2 + \cdots$$

Example

$$\dot{x} = -Lx,\ x(0)$$

$$\begin{aligned} x(t) &= \mathrm{e}^{-Lt}x(0) \\ &= \mathrm{e}^{VJV^{-1}t}x(0) \\ &= V\mathrm{e}^{Jt}V^{-1}x(0) \end{aligned}$$

matrix exponential:

$$\mathrm{e}^{VJV^{-1}t} = I + (VJV^{-1}t) + \frac{1}{2!}(VJV^{-1}t)^2 + \cdots$$

Example

$$\dot{x} = -Lx,\ x(0)$$

$$x(t) = \mathrm{e}^{-Lt}x(0)$$

$$= \mathrm{e}^{VJV^{-1}t}x(0)$$

$$= V\mathrm{e}^{Jt}V^{-1}x(0)$$

$$= [v_1\ v_2\ v_3\ v_4\ v_5] \begin{bmatrix} \mathrm{e}^{0t} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2t} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{j})t} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}-\frac{\sqrt{3}}{2}\mathrm{j})t} \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0)$$

Example

$$\dot{x} = -Lx,\ x(0)$$

$$x(t) = \mathrm{e}^{-Lt}x(0)$$
$$= \mathrm{e}^{VJV^{-1}t}x(0)$$
$$= V\mathrm{e}^{Jt}V^{-1}x(0)$$

$$= [v_1\ v_2\ v_3\ v_4\ v_5]\begin{bmatrix} \mathrm{e}^{0t} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2t} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{j})t} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}-\frac{\sqrt{3}}{2}\mathrm{j})t} \end{bmatrix}\begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0)$$

$$\to [v_1\ v_2\ v_3\ v_4\ v_5]\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0),\ \text{as } t \to \infty$$

Example

$$\dot{x} = -Lx,\ x(0)$$

$$x(t) = \mathrm{e}^{-Lt}x(0)$$

$$= \mathrm{e}^{VJV^{-1}t}x(0)$$

$$= V\mathrm{e}^{Jt}V^{-1}x(0)$$

$$= [v_1\ v_2\ v_3\ v_4\ v_5]\begin{bmatrix} \mathrm{e}^{0t} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2t} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{j})t} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}-\frac{\sqrt{3}}{2}\mathrm{j})t} \end{bmatrix}\begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0)$$

$$\to v_1 w_1^\top x(0) \qquad (v_1 = \mathbf{1},\ w_1 = [\tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ 0\ \ 0]^\top)$$

Example

$$\dot{x} = -Lx,\ x(0)$$

$$x(t) = \mathrm{e}^{-Lt}x(0)$$

$$= \mathrm{e}^{VJV^{-1}t}x(0)$$

$$= V\mathrm{e}^{Jt}V^{-1}x(0)$$

$$= [v_1\ v_2\ v_3\ v_4\ v_5]\begin{bmatrix} \mathrm{e}^{0t} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-2t} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{j})t} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{(-\frac{3}{2}-\frac{\sqrt{3}}{2}\mathrm{j})t} \end{bmatrix}\begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0)$$

$$\to v_1 w_1^\top x(0) \qquad (v_1 = \mathbf{1},\ w_1 = [\tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ 0\ \ 0]^\top)$$

$$= \left([\frac{1}{3}\ \ \frac{1}{3}\ \ \frac{1}{3}\ \ 0\ \ 0]x(0)\right)\mathbf{1}$$

Theorem

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}$$

$\dot{x} = -Lx$ solves consensus

s.t. $(\forall v_i \in \mathcal{V})(\forall x_i(0))(\exists c) \lim_{t \to \infty} x_i(t) = c$

iff $\mathcal{G}$ contains a spanning tree

Proof: (only if; necessity)

leave it as an exercise

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ contains a spanning tree

show $\dot{x} = -Lx$ solves consensus

(i) $-L$ has a simple eigenvalue 0
with eigenvector $\mathbf{1}$

hint: spanning tree $\Rightarrow$ $\text{rank}(L) = n - 1$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ contains a spanning tree

show $\dot{x} = -Lx$ solves consensus

(ii) all the other $n-1$ nonzero eigenvalues of $-L$ have negative real parts

hint: Gershgorin discs for

$$-L = -\text{diag}(A\mathbf{1}) + A$$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ contains a spanning tree

show $\dot{x} = -Lx$ solves consensus

(iii) $x(t) = \mathrm{e}^{-Lt}x(0) = \mathrm{e}^{VJV^{-1}t}x(0)$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ contains a spanning tree

show $\dot{x} = -Lx$ solves consensus

$$\text{(iii) } x(t) = \mathrm{e}^{-Lt}x(0) = \mathrm{e}^{VJV^{-1}t}x(0)$$
$$= V\mathrm{e}^{Jt}V^{-1}x(0)$$
$$= [v_1 \ v_2 \cdots v_n] \begin{bmatrix} \mathrm{e}^{0t} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{\lambda_n t} \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ \vdots \\ w_n^\top \end{bmatrix} x(0)$$

$(v_1 = \mathbf{1},\ w_1^\top v_1 = 1$ (w_1 left eigenvector of eigenvalue 0))

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ contains a spanning tree

show $\dot{x} = -Lx$ solves consensus

$$\text{(iii) } x(t) = \mathrm{e}^{-Lt}x(0) = \mathrm{e}^{VJV^{-1}t}x(0) \\ = V\mathrm{e}^{Jt}V^{-1}x(0)$$

$$(t \to \infty) \\ \to [v_1 \ v_2 \cdots v_n] \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ \vdots \\ w_n^\top \end{bmatrix} x(0)$$

$(v_1 = \mathbf{1},\ w_1^\top v_1 = 1$ (w_1 left eigenvector of eigenvalue 0))

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($v_1 = \mathbf{1}$, $w_1^\top v_1 = 1$ (w_1 left eigenvector of eigenvalue 0))

$$= v_1 w_1^\top x(0) = (w_1^\top x(0))\mathbf{1}$$

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$\dot{x} = -Lx$ solves consensus

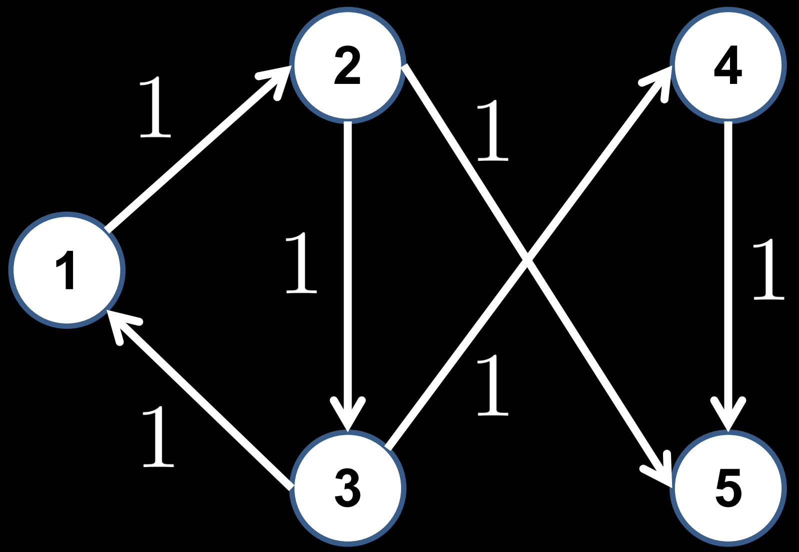
s.t. $(\forall v_i \in \mathcal{V})(\forall x_i(0))(\exists c) \lim_{t\to\infty} x_i(t) = c$

iff $\mathcal{G}$ contains a spanning tree

consensus value $c = w_1^\top x(0)$

where $w_1^\top L = 0$ and $w_1^\top \mathbf{1} = 1$

Example



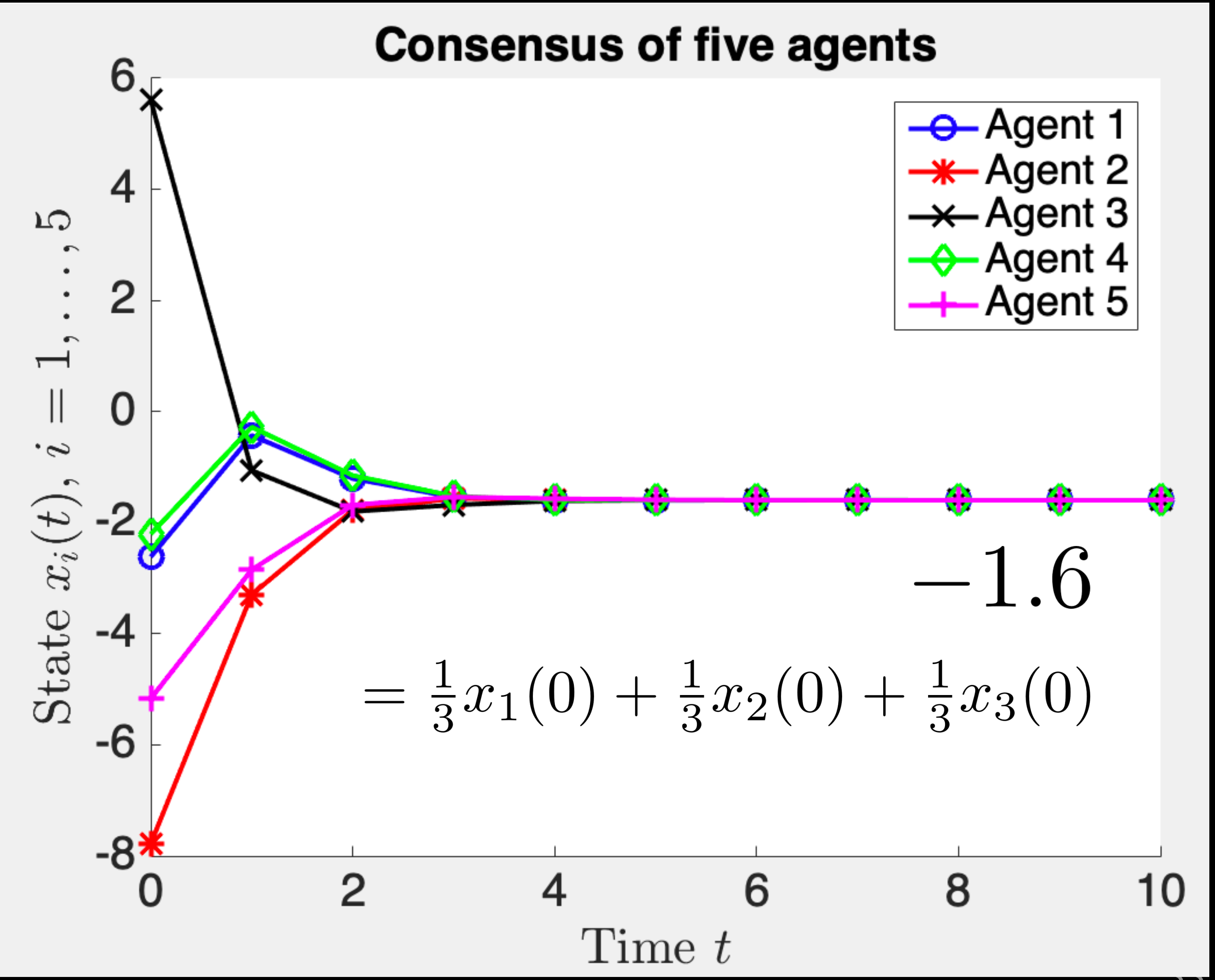
$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

$$\underbrace{[\tfrac{1}{3} \quad \tfrac{1}{3} \quad \tfrac{1}{3} \quad 0 \quad 0]}_{w_1^\top} L = 0$$

$w_1^\top$

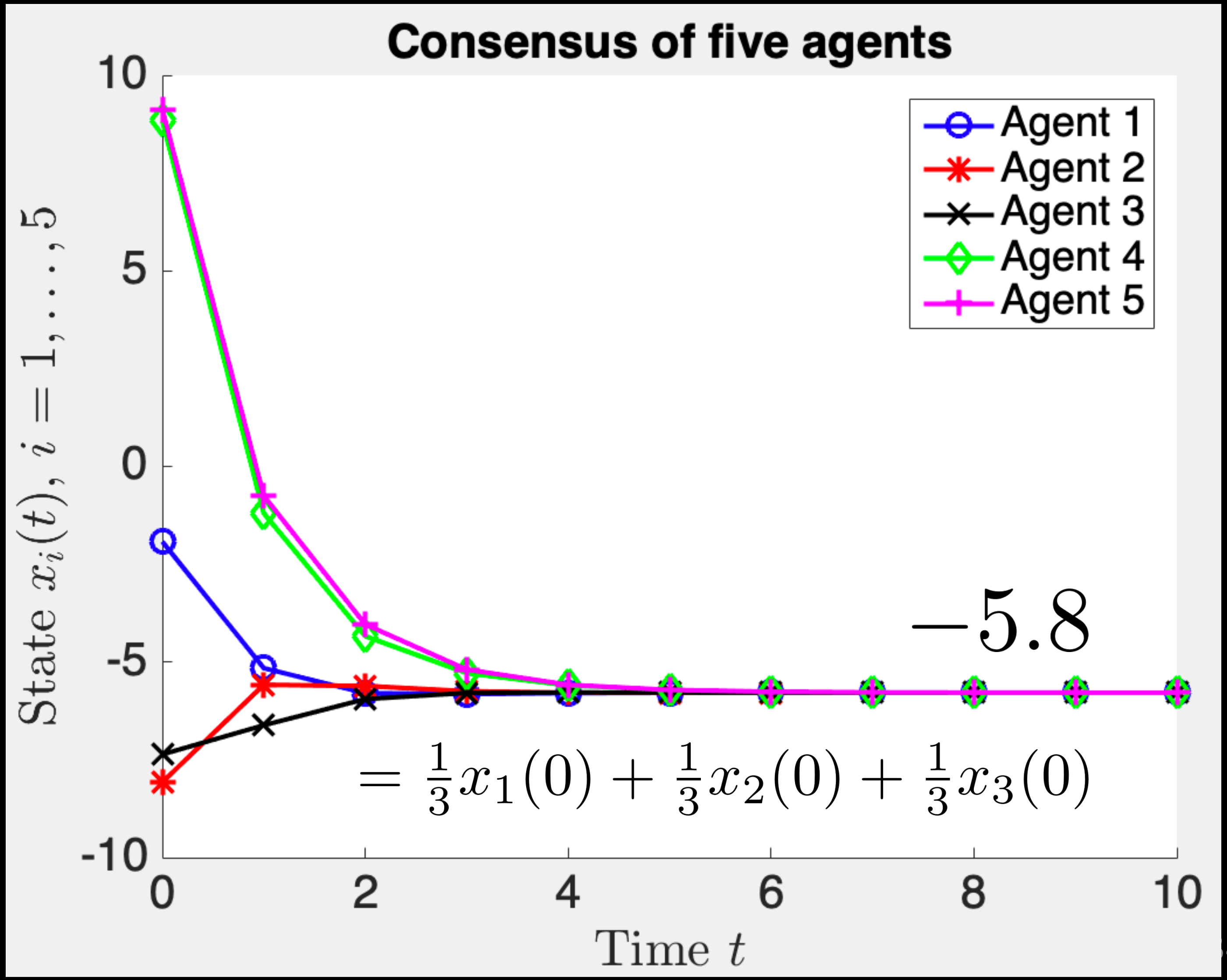
Example

simulation: $x_1(0) = -2.6, x_2(0) = -7.8$
$x_3(0) = 5.6, x_4(0) = 2.2, x_5(0) = -5.2$

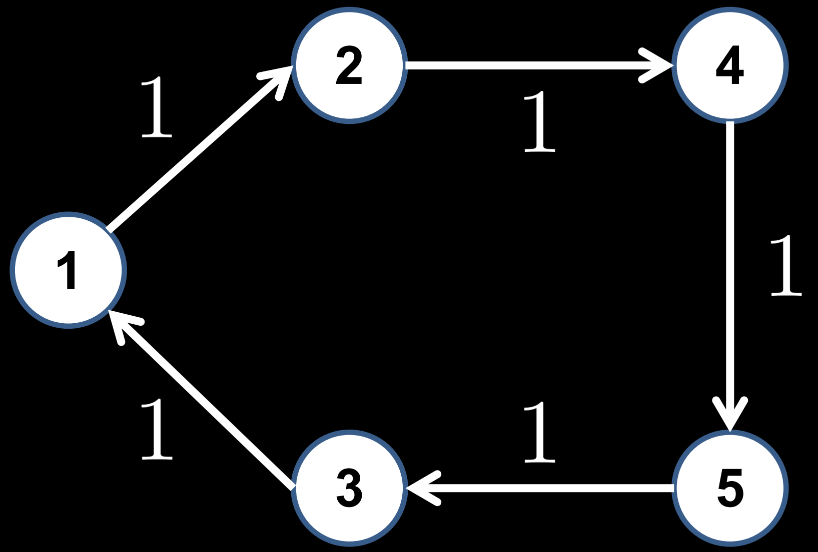


Example

$$\text{simulation: } x_1(0) = -1.9, x_2(0) = -8.1$$
$$x_3(0) = -7.4, x_4(0) = 8.8, x_5(0) = 9.1$$



Example



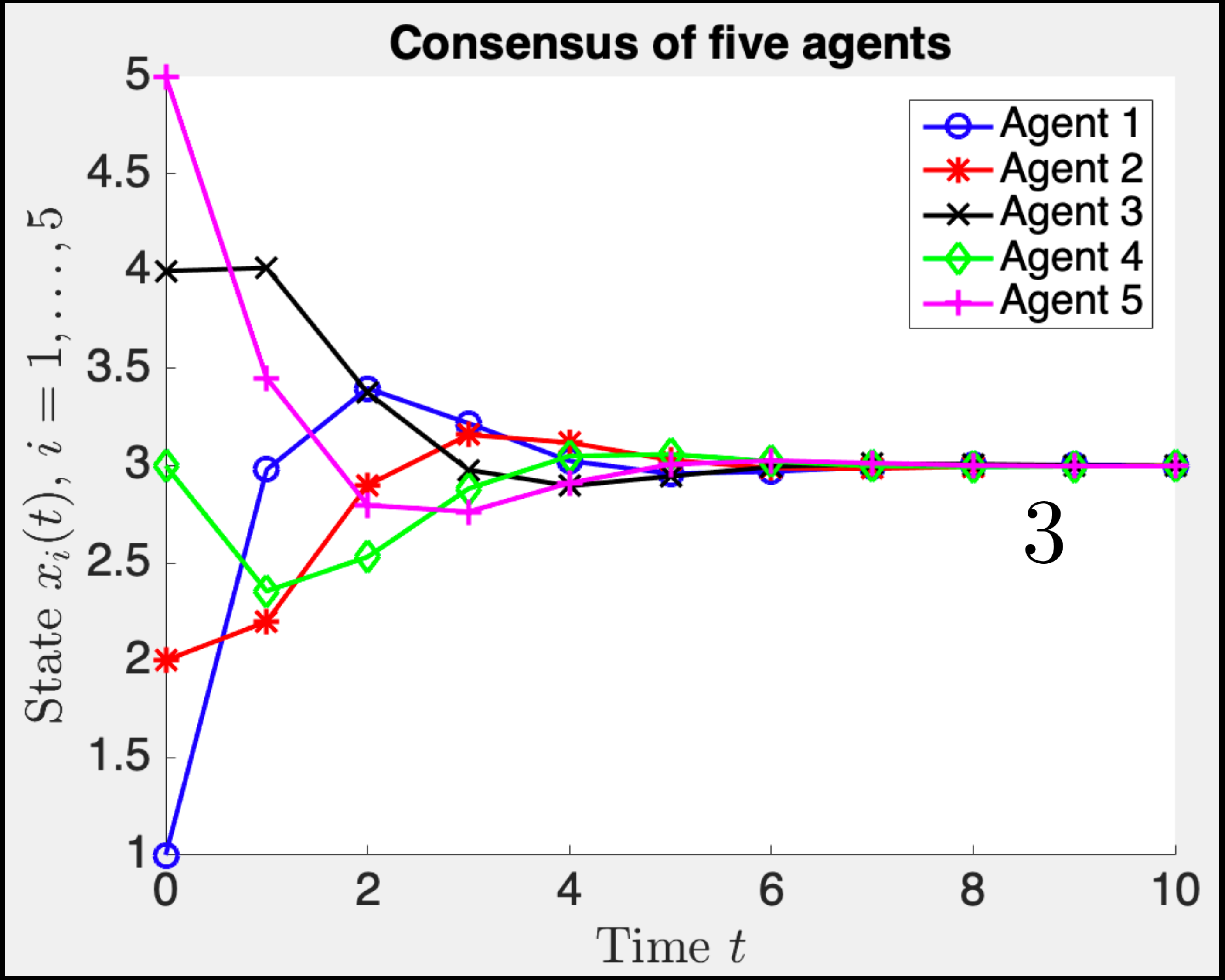
$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$[\tfrac{1}{5} \quad \tfrac{1}{5} \quad \tfrac{1}{5} \quad \tfrac{1}{5} \quad \tfrac{1}{5}]L = 0$$

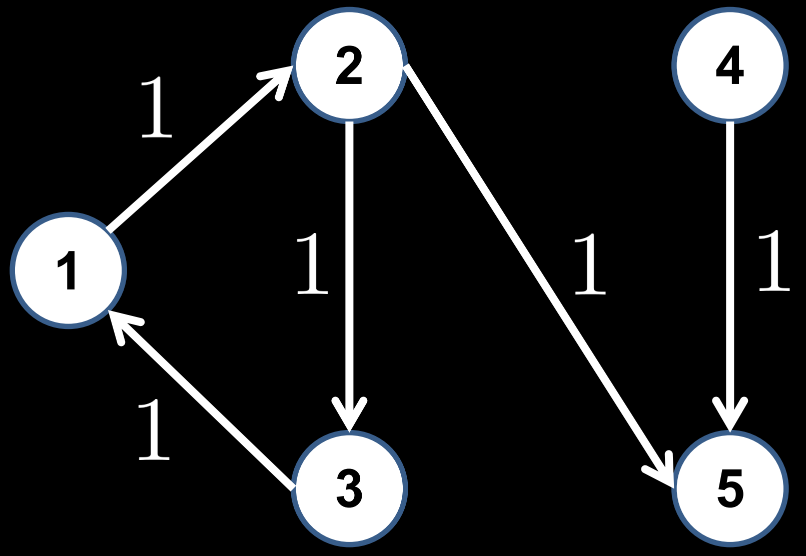
$$w_1^\top$$

Example

$$\text{simulation: } x_1(0) = 1, x_2(0) = 2$$
$$x_3(0) = 4, x_4(0) = 3, x_5(0) = 5$$



Example

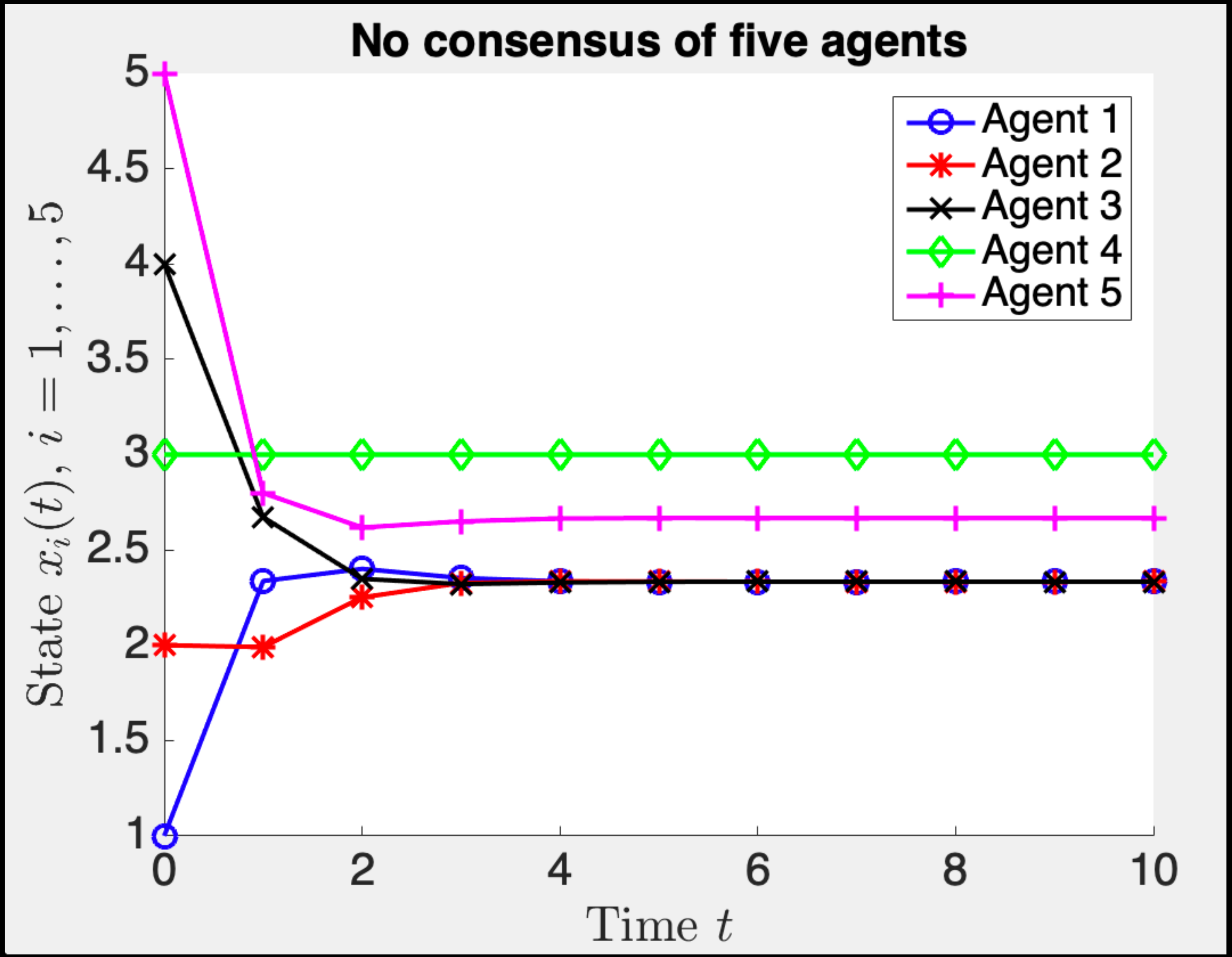


$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

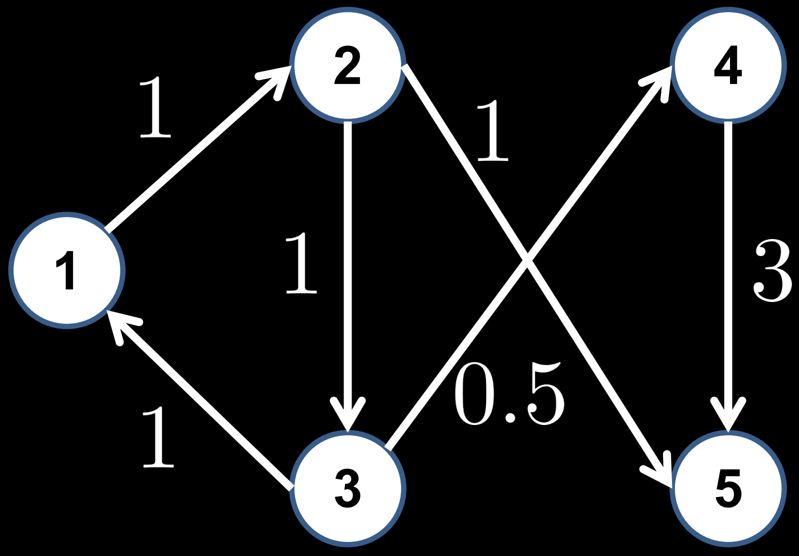
no spanning tree

Example

$$\text{simulation: } x_1(0) = 1, x_2(0) = 2$$
$$x_3(0) = 4, x_4(0) = 3, x_5(0) = 5$$



Example



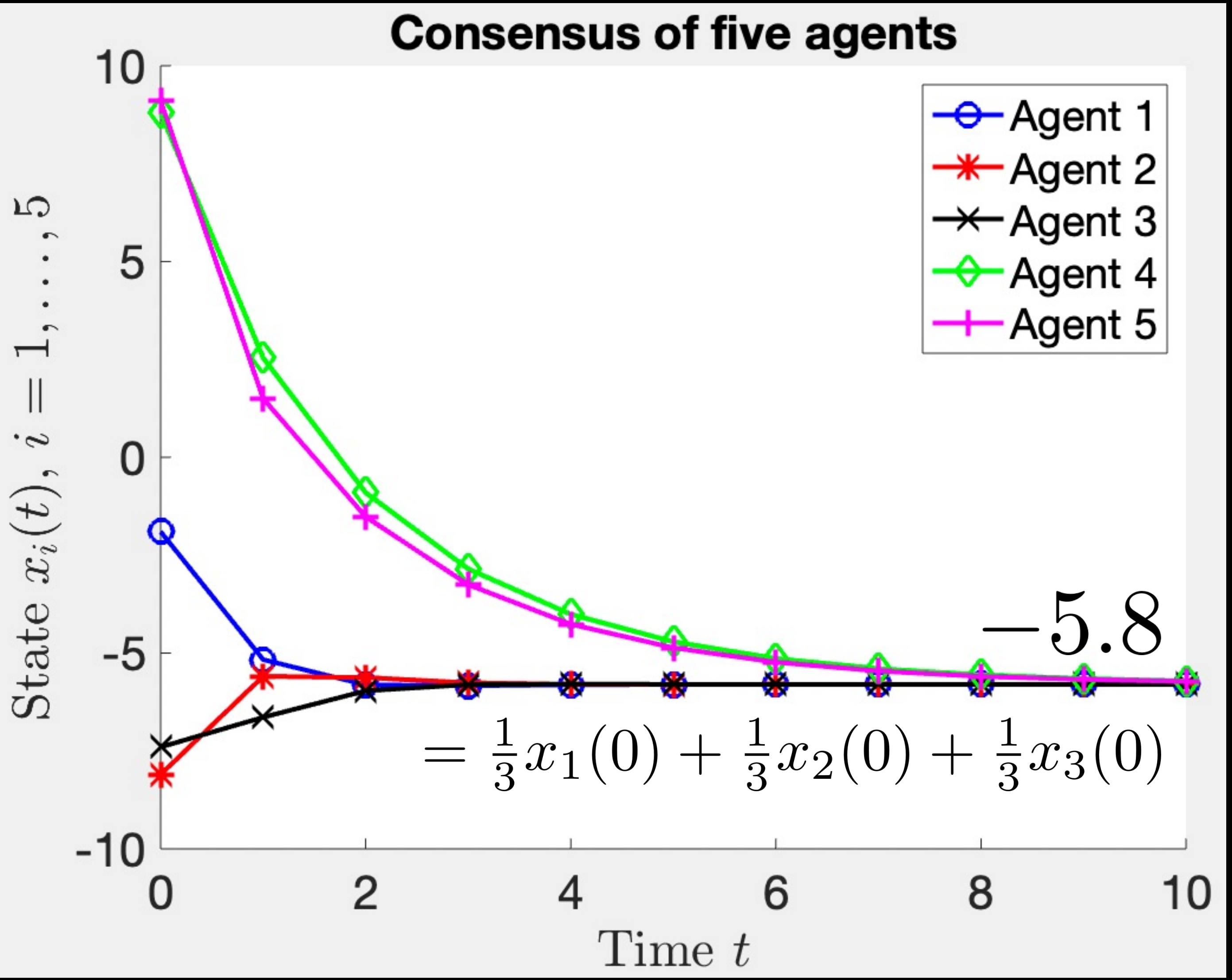
$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -0.5 & 0.5 & 0 \\ 0 & -1 & 0 & -3 & 4 \end{bmatrix}$$

$$[\tfrac{1}{3} \quad \tfrac{1}{3} \quad \tfrac{1}{3} \quad 0 \quad 0] L = 0$$

$$w_1^\top$$

Example

simulation: $x_1(0) = -1.9, x_2(0) = -8.1$
$x_3(0) = -7.4, x_4(0) = 8.8, x_5(0) = 9.1$



Different weights

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \; x_i, u_i \in \mathbb{R}$$

Distributed algorithm

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i), a_{ij} > 0$$

$\dot{x} = -Lx$ solves consensus

s.t. $(\forall v_i \in \mathcal{V})(\forall x_i(0))(\exists c) \lim_{t \to \infty} x_i(t) = c$

iff $\mathcal{G}$ contains a spanning tree