Multi-Agent Systems

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Similar Formation in 2D

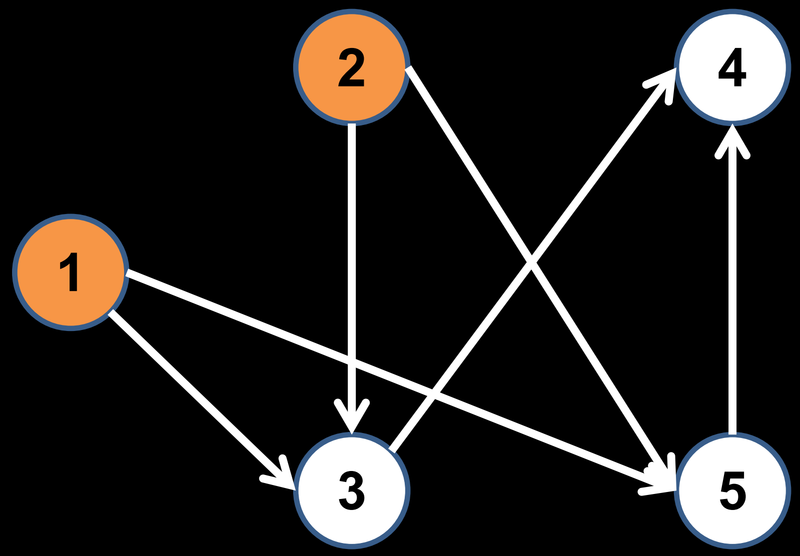
Complex Laplacian matrix of directed graph

Multi-agent system

a system of n interacting agents is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

node $v_i \in \mathcal{V}$: an agent
edge $(v_j, v_i) \in \mathcal{E}$: agent j sends information to v_i

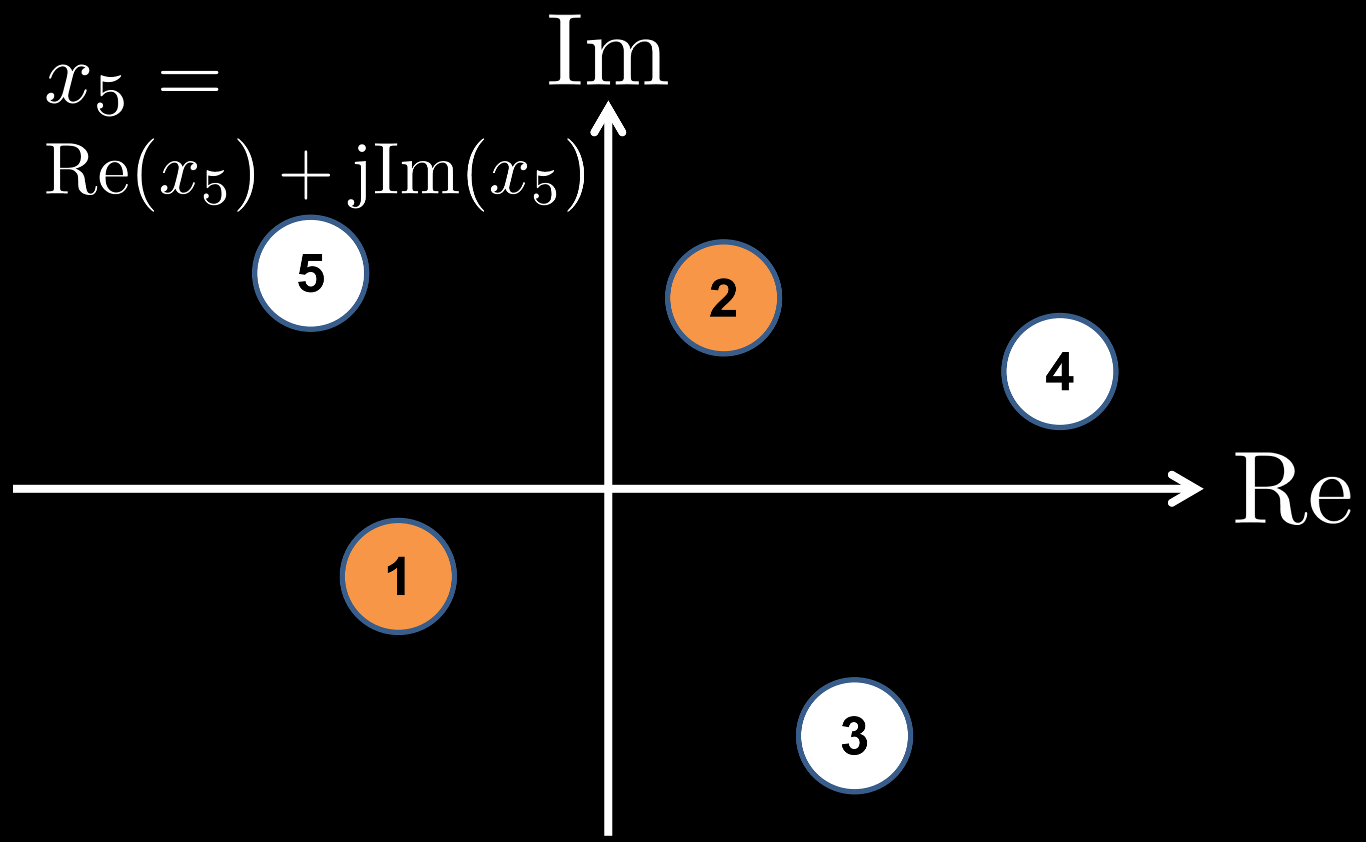
example:



2D formation problem

agents are moving on a 2D plane

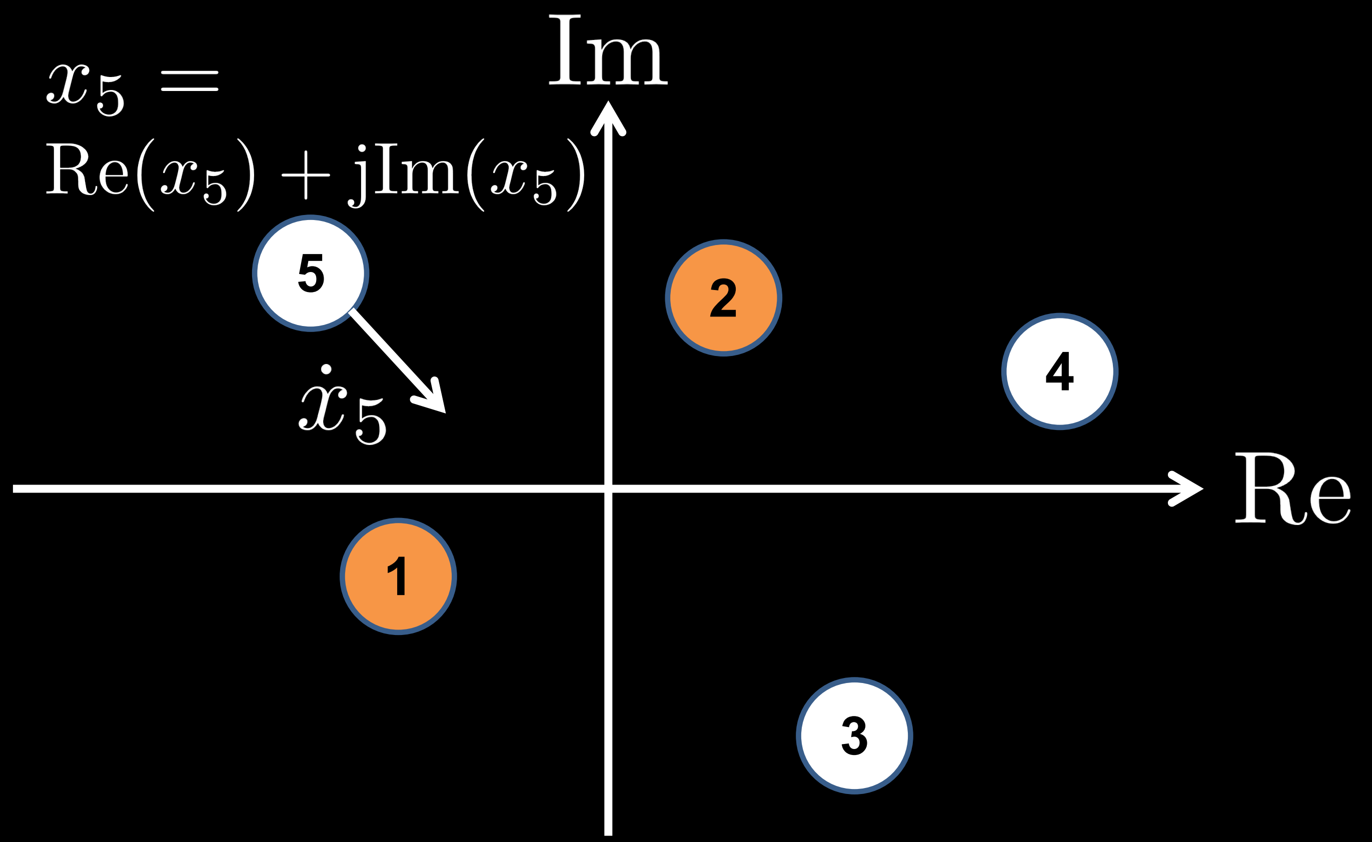
agent v_i's position is $x_i \in \mathbb{C}$



2D formation problem

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$



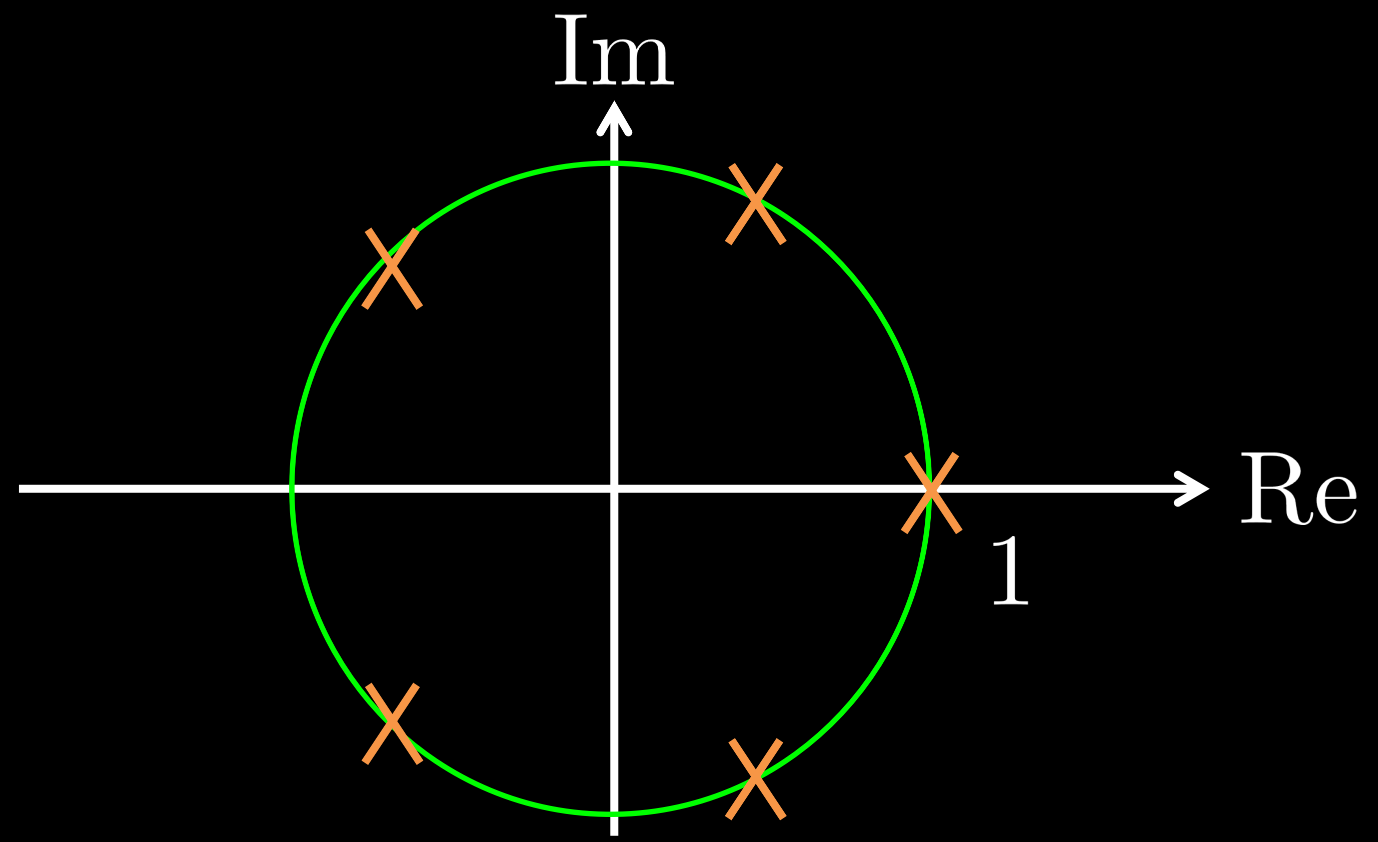
2D formation problem

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \quad x_i, u_i \in \mathbb{C}$$

target configuration:

$$\xi = [1 \quad \mathrm{e}^{\frac{2\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{4\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{6\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{8\pi}{5}\mathrm{j}}]^\top$$



2D formation problem

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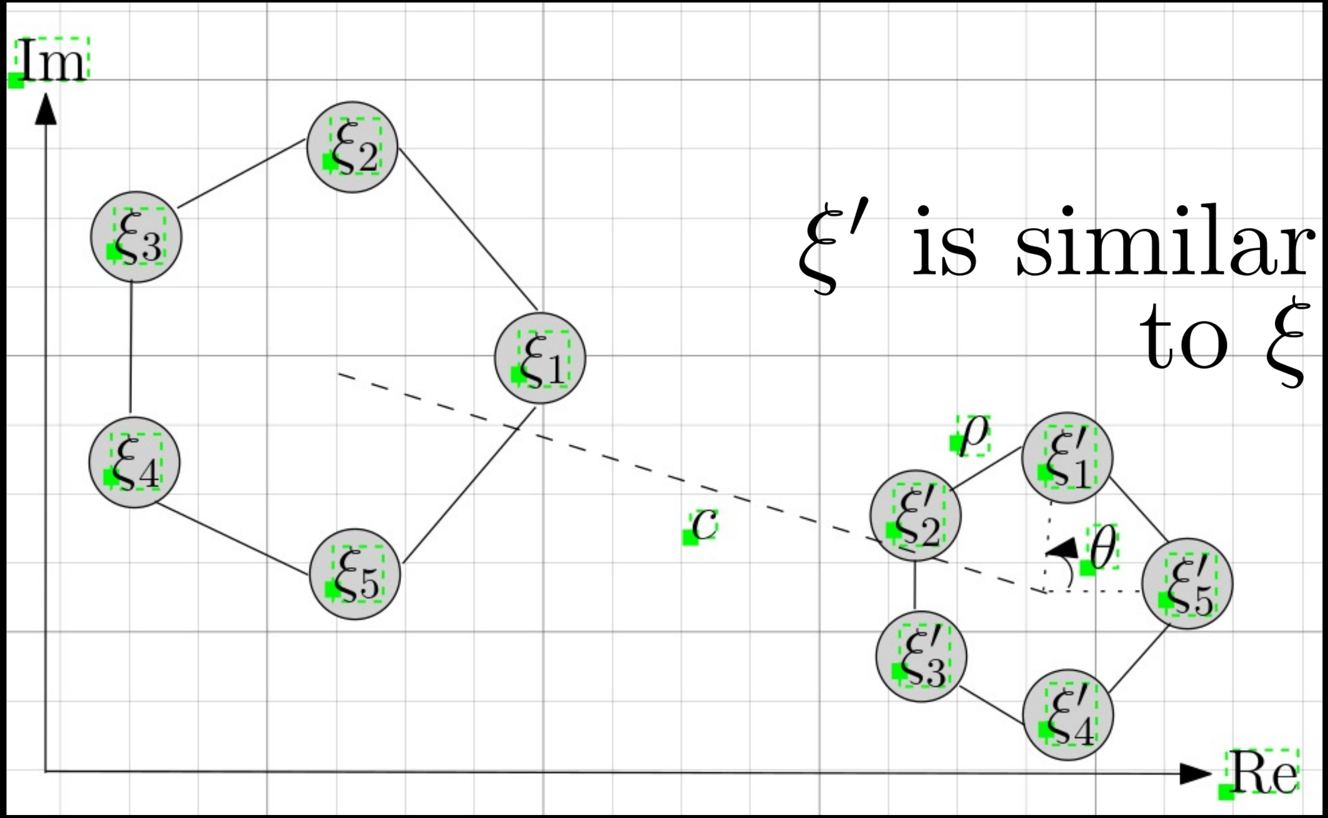


2D formation problem

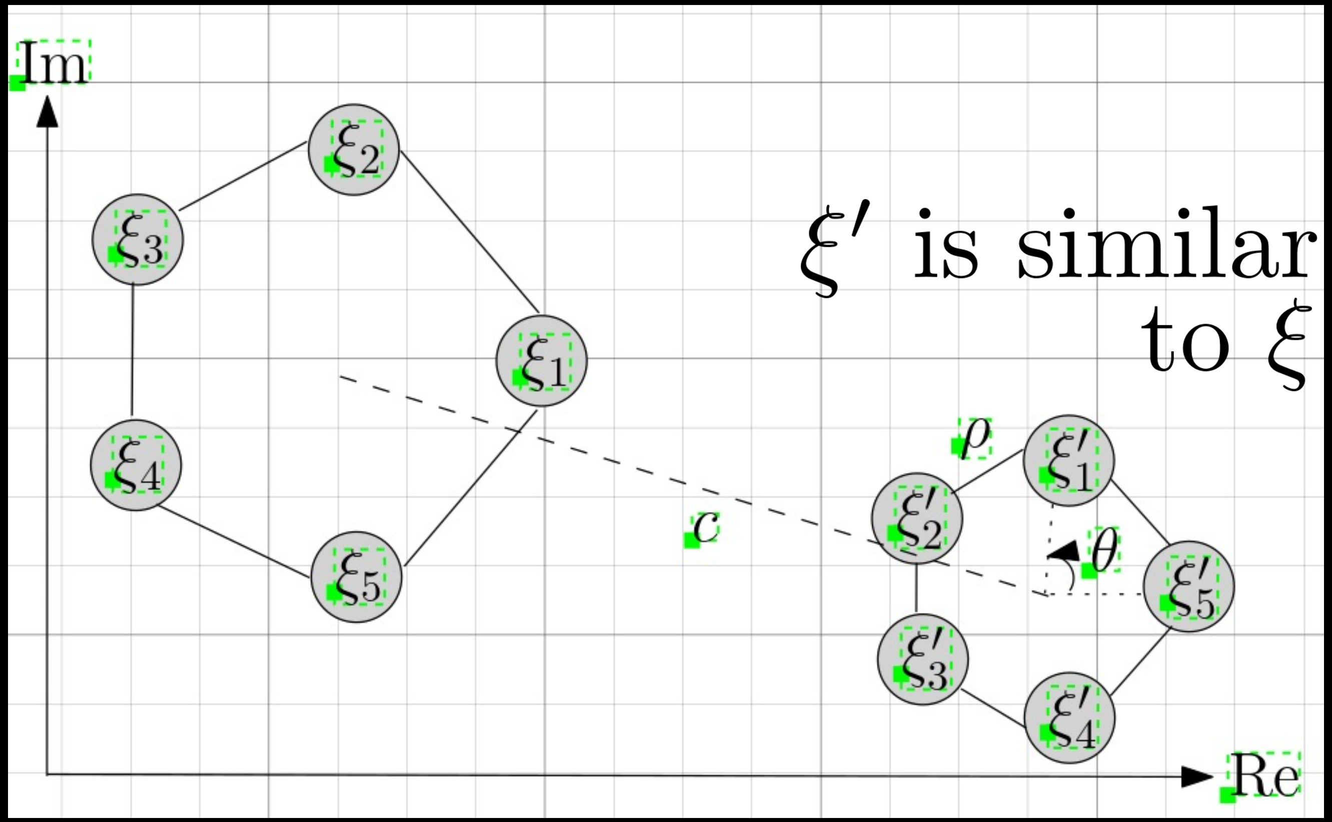
each agent v_i updates its state based on

$$\dot{x}_i = u_i,\ \ x_i, u_i \in \mathbb{C}$$

2D similar formation: design input u_i

s.t. $(\forall x_i(0))(\exists c, c' \in \mathbb{C})\ x(t) \to c\mathbf{1} + c'\xi$

$c' \downarrow \rho e^{j\theta}$



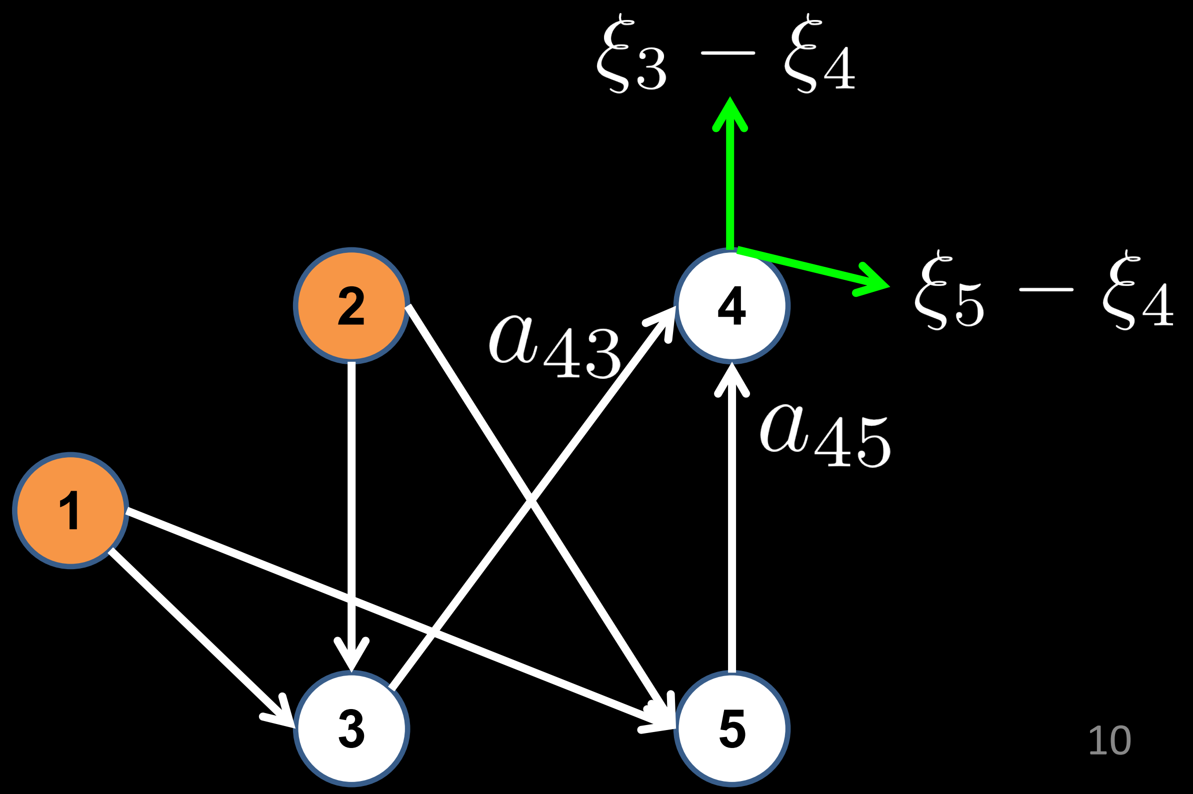
Formation constraint

target configuration:

$$\xi = [1 \quad \mathrm{e}^{\frac{2\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{4\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{6\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{8\pi}{5}\mathrm{j}}]^\top$$

$$\xi_3 - \xi_4 = 1.1756\mathrm{j}$$
$$\xi_5 - \xi_4 = 1.118 - 0.3633\mathrm{j}$$

example:



Formation constraint

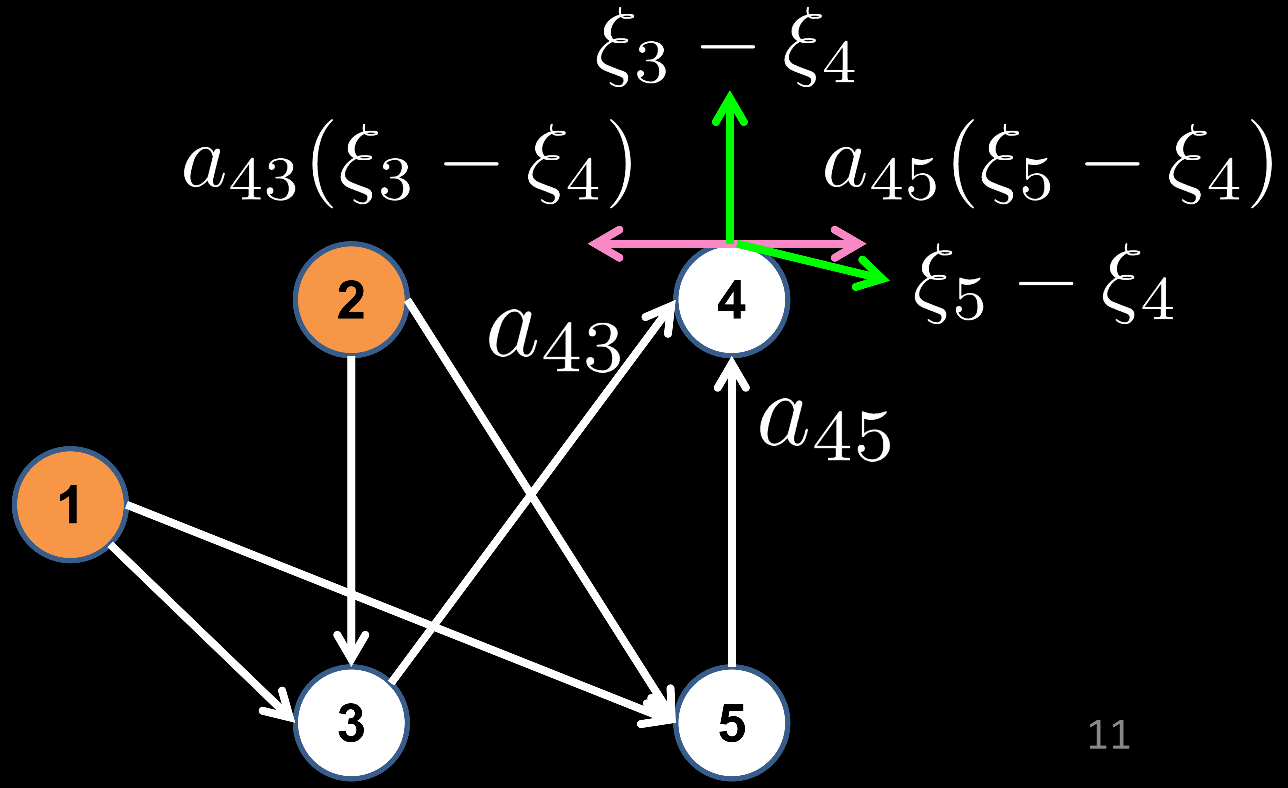
target configuration:

$$\xi = [1 \;\; e^{\frac{2\pi}{5}j} \;\; e^{\frac{4\pi}{5}j} \;\; e^{\frac{6\pi}{5}j} \;\; e^{\frac{8\pi}{5}j}]^\top$$

formation constraint:

$$a_{43}(\xi_3 - \xi_4) + a_{45}(\xi_5 - \xi_4) = 0$$

example:



Formation constraint

target configuration:

$$\xi = [1 \quad e^{\frac{2\pi}{5}j} \quad e^{\frac{4\pi}{5}j} \quad e^{\frac{6\pi}{5}j} \quad e^{\frac{8\pi}{5}j}]^\top$$

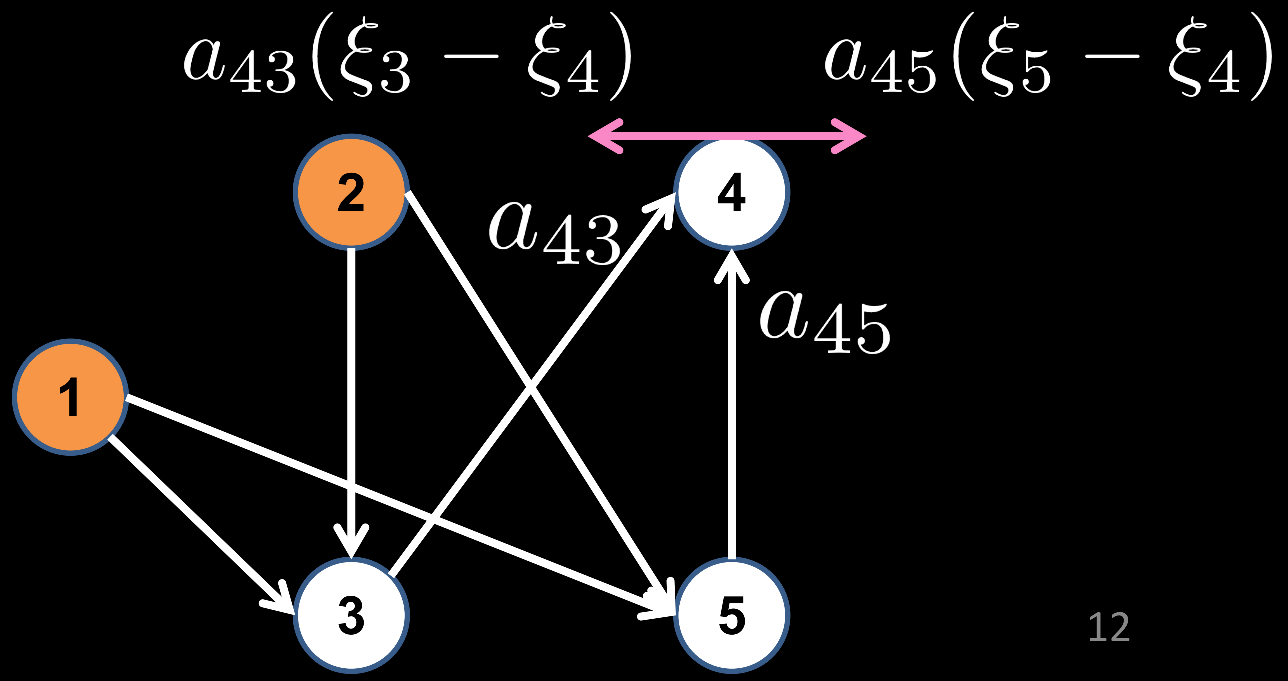
formation constraint:

$$a_{43}(\xi_3 - \xi_4) + a_{45}(\xi_5 - \xi_4) = 0$$

$$a_{43} = \frac{1}{|\xi_3 - \xi_4|} e^{(\pi - \angle \xi_3 - \xi_4)j}$$

$$a_{45} = \frac{1}{|\xi_5 - \xi_4|} e^{(-\angle \xi_5 - \xi_4)j}$$

example:



Formation constraint

target configuration:

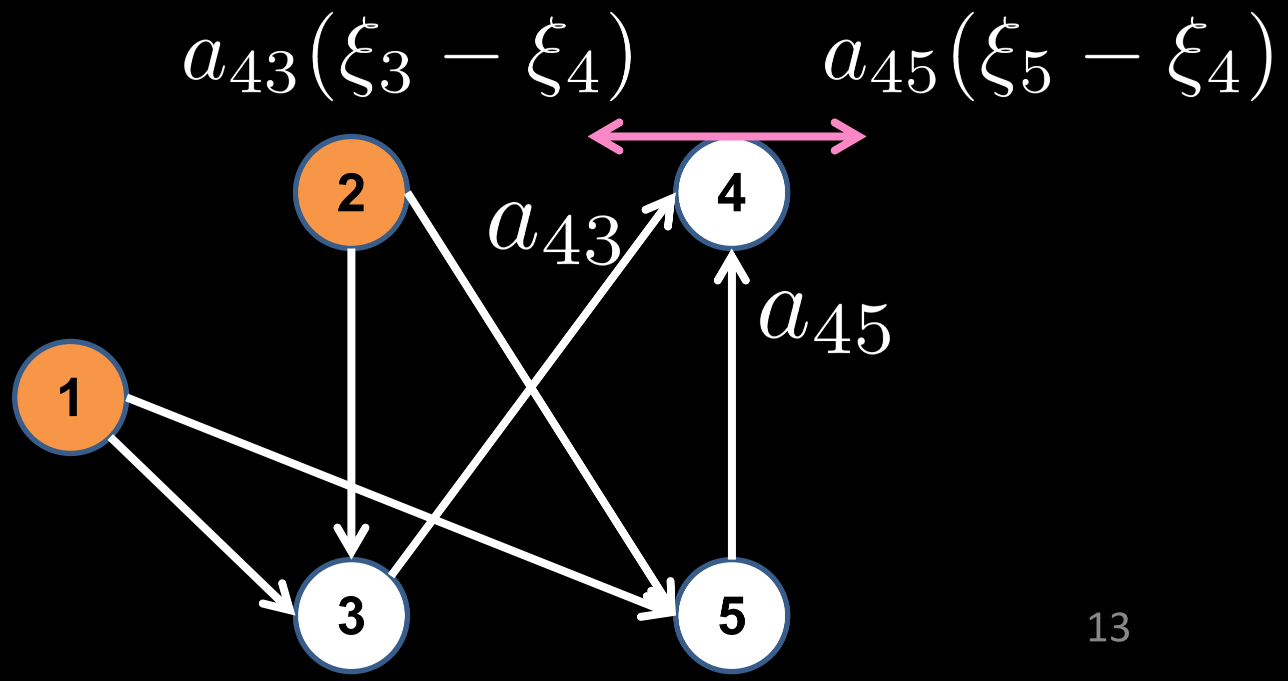
$$\xi = [1 \quad e^{\frac{2\pi}{5}\mathrm{j}} \quad e^{\frac{4\pi}{5}\mathrm{j}} \quad e^{\frac{6\pi}{5}\mathrm{j}} \quad e^{\frac{8\pi}{5}\mathrm{j}}]^\top$$

formation constraint:

$$a_{43}(\xi_3 - \xi_4) + a_{45}(\xi_5 - \xi_4) = 0$$

$$\textstyle\sum_{j \in \mathcal{N}_4} a_{4j}(\xi_j - \xi_4) = 0$$

example:



Formation constraint

target configuration:

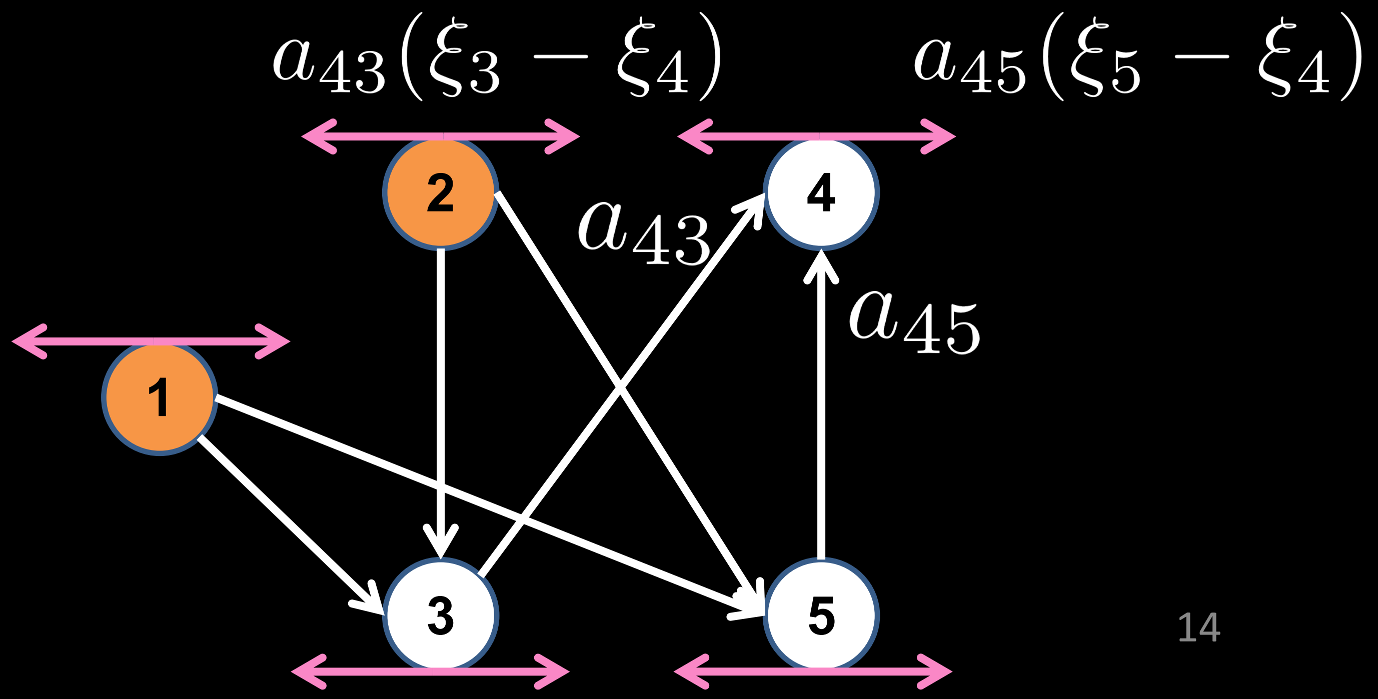
$$\xi = [1 \quad \mathrm{e}^{\frac{2\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{4\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{6\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{8\pi}{5}\mathrm{j}}]^\top$$

formation constraint:

$$(\forall i = 1, \ldots, n) \sum_{j \in \mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0$$

$$\boxed{L\xi = 0}$$

example:



Complex Laplacian

complex adjacency matrix:

$$A = [a_{ij}]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 - 0.1625\mathrm{j} & 0.809 - 0.2629\mathrm{j} & 0 & 0 & 0 \\ 0 & 0 & 0.8507\mathrm{j} & 0 & 0.809 + 0.2629\mathrm{j} \\ -0.5 + 0.6882\mathrm{j} & -0.5257\mathrm{j} & 0 & 0 & 0 \end{bmatrix}$$

complex degree matrix:

$$D = \mathrm{diag}(A\mathbf{1})$$

complex Laplacian:

$$L = D - A \qquad\qquad (L\mathbf{1} = 0)$$

Complex Laplacian

target configuration:

$$\xi = [1 \quad \mathrm{e}^{\frac{2\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{4\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{6\pi}{5}\mathrm{j}} \quad \mathrm{e}^{\frac{8\pi}{5}\mathrm{j}}]^\top$$

complex Laplacian:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 - 0.1625\mathrm{j} & 0.809 - 0.2629\mathrm{j} & -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0 & 0 & 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ -0.5 + 0.6882\mathrm{j} & -0.5257\mathrm{j} & 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

$$L\xi = 0$$

$$L\mathbf{1} = 0$$

$$\Rightarrow \mathrm{rank}(L) \leq 3$$

$$\mathrm{rank}(L) = 3(?)$$

Distributed algorithm

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

distributed algorithm:

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i)$$

relative state information

(encoding target configuration)

Distributed algorithm

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

distributed algorithm:

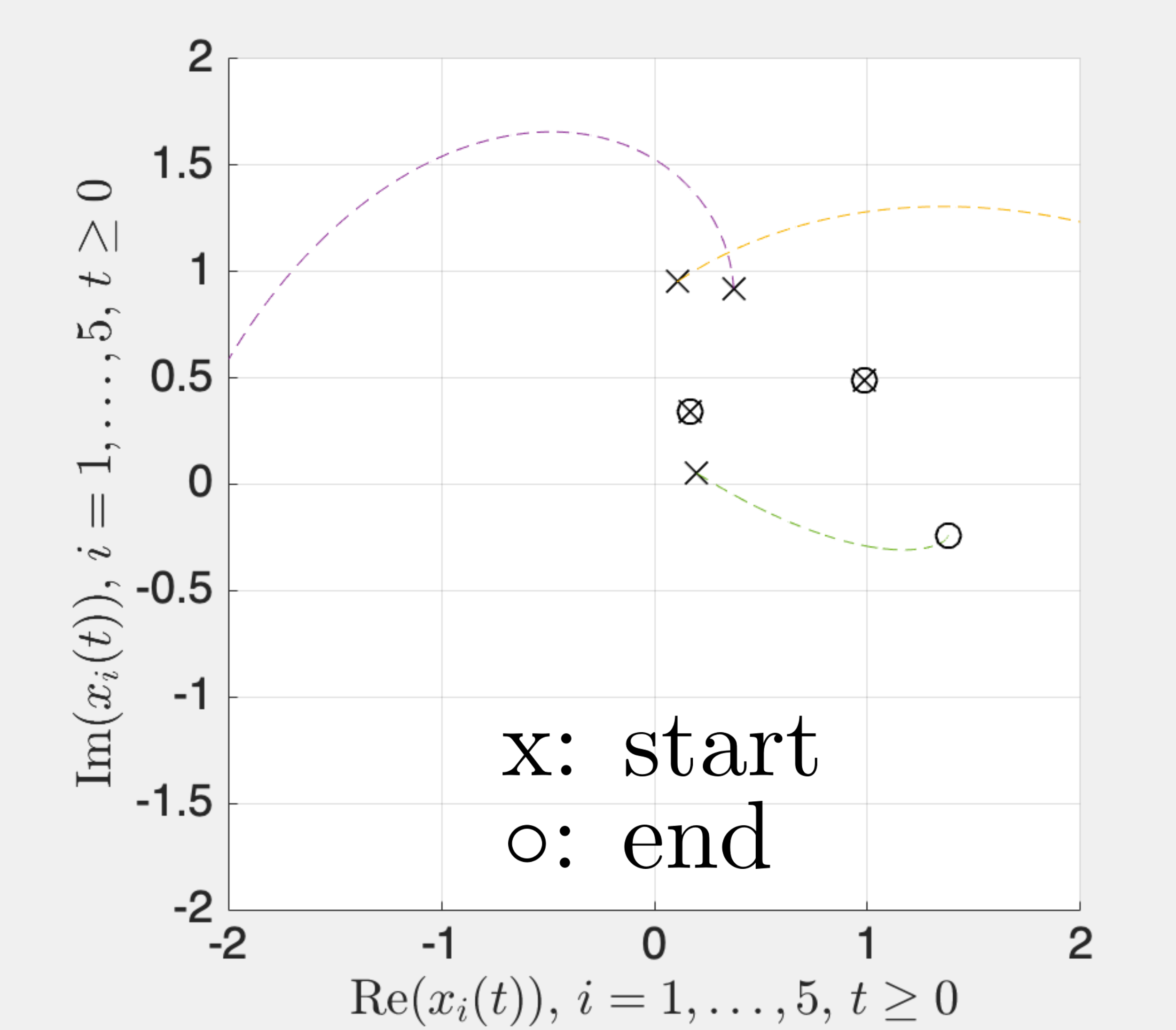
$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i)$$

$$\dot{x} = -Lx$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 - 0.1625\mathrm{j} & 0.809 - 0.2629\mathrm{j} & -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0 & 0 & 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ -0.5 + 0.6882\mathrm{j} & -0.5257\mathrm{j} & 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

Example

simulation: $x_1(0) = 0.98 + 0.49\mathrm{j}$
$x_2(0) = 0.16 + 0.34\mathrm{j}, x_3(0) = 0.11 + 0.95\mathrm{j}$
$x_4(0) = 0.37 + 0.92\mathrm{j}, x_5(0) = 0.2 + 0.05\mathrm{j}$



Example

$$\dot{x} = -Lx$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5-0.1625\mathrm{j} & 0.809-0.2629\mathrm{j} & -0.309+0.4253\mathrm{j} & 0 & 0 \\ 0 & 0 & 0.8507\mathrm{j} & -0.809-1.1135\mathrm{j} & 0.809+0.2629\mathrm{j} \\ -0.5+0.6882\mathrm{j} & -0.5257\mathrm{j} & 0 & 0 & 0.5-0.1625\mathrm{j} \end{bmatrix}$$

eigenvalues of $-L$:

$$0, 0, -0.5+0.16\mathrm{j}, 0.31-0.43\mathrm{j}, 0.81+1.11\mathrm{j}$$

design invertible diagonal matrix E s.t. nonzero eigenvalues of $-EL$ have negative real parts

Distributed algorithm

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

distributed algorithm:

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} \epsilon_i a_{ij} (x_j - x_i), \; \epsilon_i, a_{ij} \in \mathbb{C}$$

(ensuring stability) (encoding target configuration)

$\dot{x} = -ELx$, where $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$

Example

$$\dot{x} = -ELx$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 - 0.1625\mathrm{j} & 0.809 - 0.2629\mathrm{j} & -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0 & 0 & 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ -0.5 + 0.6882\mathrm{j} & -0.5257\mathrm{j} & 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 0$$

$$L' = \begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

$$\text{design } E' = \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \text{ s.t. all eigenvalues of } -E'L' \text{ are stable}$$

equivalently, all eigenvalues of $E'L'$ have positive real parts $(\lambda_3, \lambda_4, \lambda_5)$

Example

$$\begin{array}{ll} & \quad L_1' \\ L' = & \begin{bmatrix} \boxed{-0.309 + 0.4253\mathrm{j}} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix} \end{array}$$

$$\begin{array}{ll} & E_1' \\ \text{design } E' = & \begin{bmatrix} \boxed{\epsilon_3} & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \text{ s.t. all eigenvalues of } -E'L' \text{ are stable} \end{array}$$

equivalently, all eigenvalues of $E'L'$ have positive real parts

$$(\boxed{\lambda_3,}\, \lambda_4, \lambda_5)$$

Example

$$\det(E_1' L_1') = \lambda_3$$
$$\det(E_1')\ \det(L_1') = \lambda_3$$
$$\epsilon_3 l_1 = \lambda_3$$

$$\begin{array}{cc} & \begin{array}{ccc} L_1' & & \end{array} \\ L' = & \begin{bmatrix} \boxed{-0.309 + 0.4253\mathrm{j}} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix} \end{array}$$

$$\text{design } E' = \begin{array}{cc} E_1' & \\ & \begin{bmatrix} \boxed{\epsilon_3} & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \end{array} \text{ s.t. all eigenvalues of } -E'L' \text{ are stable}$$

equivalently, all eigenvalues of $E'L'$ have positive real parts

$$(\boxed{\lambda_3,} \lambda_4, \lambda_5)$$

Example

$$\det(E_1' L_1') = \lambda_3$$
$$\det(E_1')\,\det(L_1') = \lambda_3$$
$$\epsilon_3 l_1 = \lambda_3$$

choose ϵ_3 s.t. $\lambda_3 = \epsilon_3 l_1$ has positive real part

e.g. $\epsilon_3 = \frac{1}{|l_1|} \mathrm{e}^{-\mathrm{j}\angle l_1} = -1.1181 - 1.5389\mathrm{j}$

set $\lambda_3 := \epsilon_3 l_1 = 1$

$$L' = \begin{array}{c} L_1' \\ \begin{bmatrix} \boxed{-0.309 + 0.4253\mathrm{j}} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix} \end{array}$$

design $E' = \begin{array}{c} E_1' \\ \begin{bmatrix} \boxed{\epsilon_3} & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \end{array}$ s.t. all eigenvalues of $-E'L'$ are stable

equivalently, all eigenvalues of $E'L'$ have positive real parts

$(\boxed{\lambda_3}, \lambda_4, \lambda_5)$

Example

$$
\begin{array}{c} L_2' \\ L' = \begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix} \end{array}
$$

$$
\begin{array}{c} E_2' \\ \text{design } E' = \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \text{ s.t. all eigenvalues of } -E'L' \text{ are stable} \end{array}
$$

equivalently, all eigenvalues of $E'L'$ have positive real parts

$$(\boxed{\lambda_3, \lambda_4,} \lambda_5)$$

Example

$$\det(E_2' L_2') = \lambda_3 \lambda_4$$

$$\det(E_2')\ \det(L_2') = \lambda_3 \lambda_4$$

$$\epsilon_3 \epsilon_4 l_2 = \lambda_3 \lambda_4$$

$$l_2 = \det \begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} \end{bmatrix} = 0.7236$$

$$L_2'$$

$$L' = \begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

$$E_2'$$

$$\text{design } E' = \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \text{ s.t. all eigenvalues of } -E'L' \text{ are stable}$$

equivalently, all eigenvalues of $E'L'$ have positive real parts

$$(\lambda_3, \lambda_4, \lambda_5)$$

Example

$$\det(E_2' L_2') = \lambda_3 \lambda_4$$

$$\det(E_2')\,\det(L_2') = \lambda_3 \lambda_4$$

$$\epsilon_3 \epsilon_4 l_2 = \lambda_3 \lambda_4$$

choose ϵ_4 s.t. $\lambda_4 = \frac{\epsilon_4 \epsilon_3 l_2}{\lambda_3}$ has positive real part

e.g. $\epsilon_4 = \epsilon \mathrm{e}^{-\mathrm{j}\angle \frac{\epsilon_3 l_2}{\lambda_3}} = 0.1(-0.5878 + 0.809\mathrm{j})$

set $\lambda_4 := \frac{\epsilon_4 \epsilon_3 l_2}{\lambda_3} = 0.1376$

$$L' = \begin{array}{c} L_2' \\ \begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix} \end{array}$$

design $E' = \begin{array}{c} E_2' \\ \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \end{array}$ s.t. all eigenvalues of $-E'L'$ are stable

equivalently, all eigenvalues of $E'L'$ have positive real parts

$$(\boxed{\lambda_3, \lambda_4,}\, \lambda_5)$$

Example

$$L' = \begin{bmatrix} -0.309 + 0.4253\text{j} & 0 & 0 \\ 0.8507\text{j} & -0.809 - 1.1135\text{j} & 0.809 + 0.2629\text{j} \\ 0 & 0 & 0.5 - 0.1625\text{j} \end{bmatrix}$$

$$\text{design } E' = \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \text{ s.t. all eigenvalues of } -E'L' \text{ are stable}$$

equivalently, all eigenvalues of $E'L'$ have positive real parts

$(\lambda_3, \lambda_4, \lambda_5)$

Example

$$\det(E'L') = \lambda_3\lambda_4\lambda_5$$
$$\det(E')\,\det(L') = \lambda_3\lambda_4\lambda_5$$
$$\epsilon_3\epsilon_4\epsilon_5 l = \lambda_3\lambda_4\lambda_5$$

$$l = \det(L') = 0.3618 - 0.1176\mathrm{j}$$

$$L' = \begin{bmatrix} \boxed{\begin{matrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{matrix}} \end{bmatrix}$$

$$\text{design } E' = \begin{bmatrix} \boxed{\begin{matrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{matrix}} \end{bmatrix} \text{ s.t. all eigenvalues of } -E'L' \text{ are stable}$$

equivalently, all eigenvalues of $E'L'$ have positive real parts

$$(\boxed{\lambda_3, \lambda_4, \lambda_5})$$

Example

$$\det(E'L') = \lambda_3\lambda_4\lambda_5$$

$$\det(E')\,\det(L') = \lambda_3\lambda_4\lambda_5$$

$$\epsilon_3\epsilon_4\epsilon_5 l = \lambda_3\lambda_4\lambda_5$$

choose ϵ_5 s.t. $\lambda_5 = \frac{\epsilon_5\epsilon_4\epsilon_3 l_3}{\lambda_4\lambda_3}$ has positive real part

e.g. $\epsilon_5 = \epsilon \mathrm{e}^{-\mathrm{j}\angle \frac{\epsilon_4\epsilon_3 l_3}{\lambda_4\lambda_3}} = 0.1(0.951 + 0.3091\mathrm{j})$

set $\lambda_5 := \frac{\epsilon_5\epsilon_4\epsilon_3 l_3}{\lambda_4\lambda_3} = 0.0526$

$$L' = \boxed{\begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}}$$

design $E' = \boxed{\begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix}}$ s.t. all eigenvalues of $-E'L'$ are stable

equivalently, all eigenvalues of $E'L'$ have positive real parts

$(\boxed{\lambda_3, \lambda_4, \lambda_5})$

Example

$$L' = \begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

$$E' = \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix}$$
$$= \begin{bmatrix} -1.1181 - 1.5389\mathrm{j} & 0 & 0 \\ 0 & -0.0588 + 0.0809\mathrm{j} & 0 \\ 0 & 0 & 0.0951 + 0.0309\mathrm{j} \end{bmatrix}$$

eigenvalues of $E'L'$: $1, 0.1376, 0.0526$

Example

$$L' = \begin{bmatrix} -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

$$\begin{aligned} E' &= \begin{bmatrix} \epsilon_3 & 0 & 0 \\ 0 & \epsilon_4 & 0 \\ 0 & 0 & \epsilon_5 \end{bmatrix} \\ &= \begin{bmatrix} -1.1181 - 1.5389\mathrm{j} & 0 & 0 \\ 0 & -0.0588 + 0.0809\mathrm{j} & 0 \\ 0 & 0 & 0.0951 + 0.0309\mathrm{j} \end{bmatrix} \end{aligned}$$

eigenvalues of $E'L'$: $1, 0.1376, 0.0526$

eigenvalues of $-E'L'$: $-1, -0.1376, -0.0526$

Example

$$\dot{x} = -ELx$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 - 0.1625\text{j} & 0.809 - 0.2629\text{j} & -0.309 + 0.4253\text{j} & 0 & 0 \\ 0 & 0 & 0.8507\text{j} & -0.809 - 1.1135\text{j} & 0.809 + 0.2629\text{j} \\ -0.5 + 0.6882\text{j} & -0.5257\text{j} & 0 & 0 & 0.5 - 0.1625\text{j} \end{bmatrix}$$

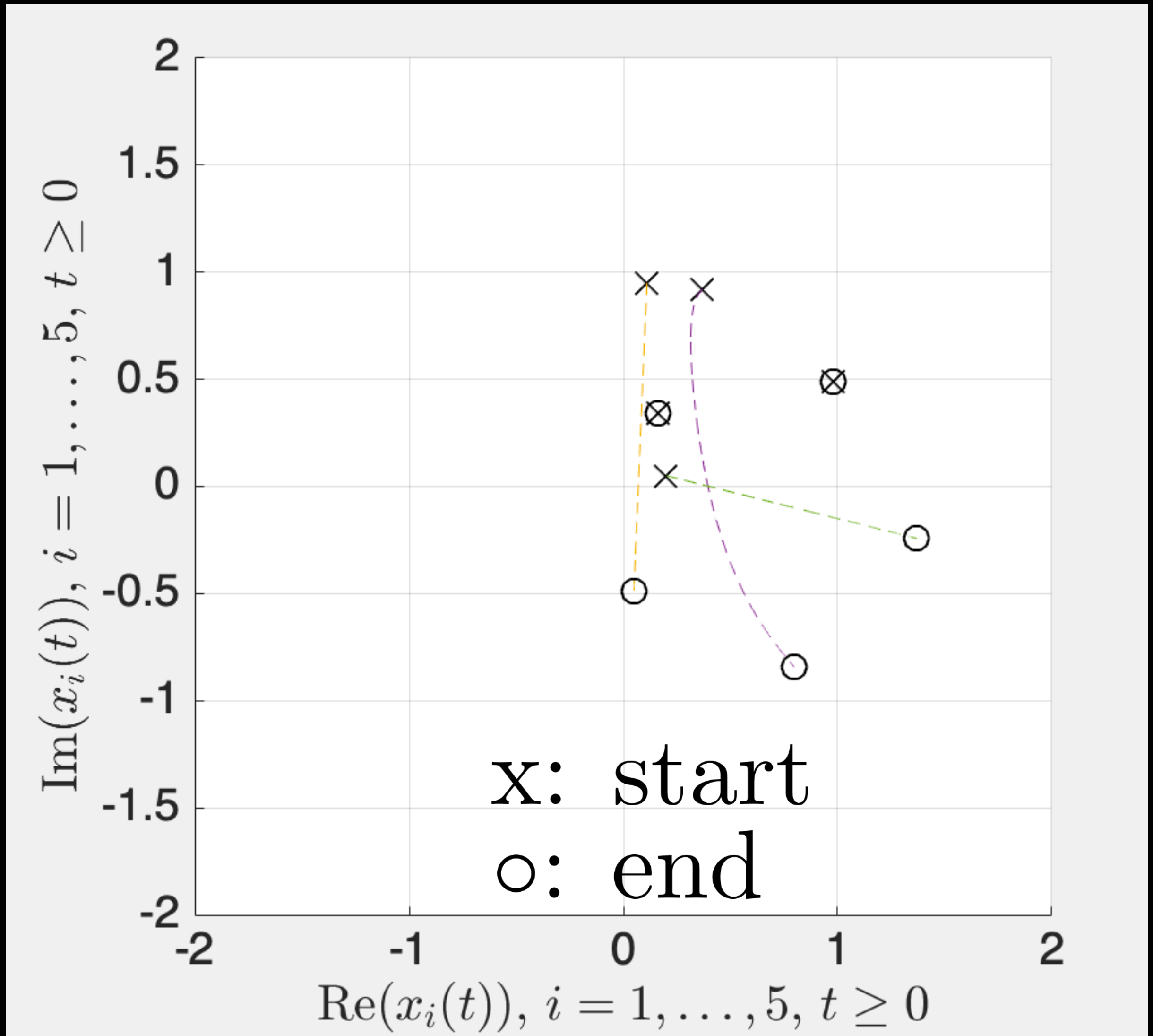
$$E = \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_4 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & -1.1181 - 1.5389\text{j} & 0 & 0 \\ 0 & 0 & 0 & -0.0588 + 0.0809\text{j} & 0 \\ 0 & 0 & 0 & 0 & 0.0951 + 0.0309\text{j} \end{bmatrix}$$

eigenvalues of $-EL$: $0, 0, -1, -0.1376, -0.0526$

Example

simulation: $x_1(0) = 0.98 + 0.49\mathrm{j}$
$x_2(0) = 0.16 + 0.34\mathrm{j}, x_3(0) = 0.11 + 0.95\mathrm{j}$
$x_4(0) = 0.37 + 0.92\mathrm{j}, x_5(0) = 0.2 + 0.05\mathrm{j}$



Fact

$$\dot{x} = -ELx$$

If $\mathcal{G}$ contains a spanning 2-tree
and ξ generic (no 3 points on same line)
then E exists s.t. $n-2$ nonzero
eigenvalues of $-EL$ are stable

[Friedland, 1975]

global computation (need L)

Generalization

a system of n interacting agents is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

target configuration $\xi = [\xi_1 \cdots \xi_n]^\top$

Problem: design u_i to update x_i

s.t. $(\forall v_i \in \mathcal{V})(\forall x_i(0))(\exists c, c' \in \mathbb{C})$

$$\lim_{t \to \infty} x(t) = c\mathbf{1} + c'\xi$$

Generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

target configuration $\xi = [\xi_1 \cdots \xi_n]^\top$

similar formation set

$$\mathcal{S}(\xi) = \{\xi' : (\exists c, c' \in \mathbb{C})\xi' = c\mathbf{1} + c'\xi\}$$

Problem: design u_i to update x_i

$$(\forall x(0) \in \mathbb{C}^n)(\exists \xi' \in \mathcal{S}(\xi)) \lim_{t \to \infty} x(t) = \xi'$$

Generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

target configuration $\xi = [\xi_1 \cdots \xi_n]^\top$

Distributed algorithm

$$\dot{x}_i = u_i = \epsilon_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i), \epsilon_i, a_{ij} \in \mathbb{C}$$

where $\sum_{j \in \mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0$

based on $x_j(t) - x_i(t)$, $\xi_j - \xi_i$
from neighbor agent(s) $j \in \mathcal{N}_i$

Generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

target configuration $\xi = [\xi_1 \cdots \xi_n]^\top$

$\dot{x} = -ELx$, where $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$

$L\mathbf{1} = 0$

$L\xi = 0$

$\ker(L) \supseteq \mathcal{S}(\xi) = \{\xi' : (\exists c, c' \in \mathbb{C})\xi' = c\mathbf{1} + c'\xi\}$

Theorem

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

target configuration $\xi = [\xi_1 \cdots \xi_n]^\top$

if ξ is generic and $\mathcal{G}$ contains
a spanning 2-tree

then $\operatorname{rank}(L) = n - 2$ and $\ker(L) = \mathcal{S}(\xi)$

and there exists E s.t. $n-2$ nonzero
eigenvalues of $-EL$ are stable

Theorem

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

target configuration $\xi = [\xi_1 \cdots \xi_n]^\top$

if ξ is generic and $\mathcal{G}$ contains
a spanning 2-tree

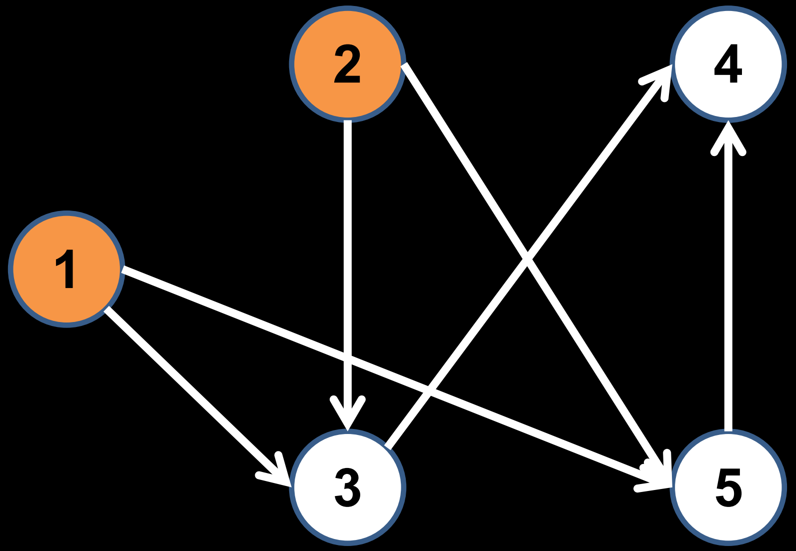
then there exists E s.t. $\dot{x} = -ELx$
solves 2D formation problem

$$(\forall x(0) \in \mathbb{C}^n)(\exists \xi' \in \mathcal{S}(\xi)) \lim_{t \to \infty} x(t) = \xi'$$

Example

example:

weighted graph $\mathcal{G}$



spanning 2-tree (?)

generic configuration (?)

$$\xi = [1 \;\; e^{\frac{2\pi}{5}j} \;\; e^{\frac{4\pi}{5}j} \;\; e^{\frac{6\pi}{5}j} \;\; e^{\frac{8\pi}{5}j}]^\top$$

Example

$$\dot{x} = -ELx$$

complex Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 - 0.1625\mathrm{j} & 0.809 - 0.2629\mathrm{j} & -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0 & 0 & 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ -0.5 + 0.6882\mathrm{j} & -0.5257\mathrm{j} & 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

$$\mathrm{rank}(L) = 3,\ \lambda_1 = 0, v_1 = \mathbf{1},\ \lambda_2 = 0, v_2 = \xi$$

so $\ker(L) = \mathcal{S}(\xi)$

Example

$$\dot{x} = -ELx$$

complex Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5 - 0.1625\mathrm{j} & 0.809 - 0.2629\mathrm{j} & -0.309 + 0.4253\mathrm{j} & 0 & 0 \\ 0 & 0 & 0.8507\mathrm{j} & -0.809 - 1.1135\mathrm{j} & 0.809 + 0.2629\mathrm{j} \\ -0.5 + 0.6882\mathrm{j} & -0.5257\mathrm{j} & 0 & 0 & 0.5 - 0.1625\mathrm{j} \end{bmatrix}$$

stabilizing diagonal matrix

$$E = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & -1.1181 - 1.5389\mathrm{j} & 0 & 0 \\ 0 & 0 & 0 & -0.0588 + 0.0809\mathrm{j} & 0 \\ 0 & 0 & 0 & 0 & 0.0951 + 0.0309\mathrm{j} \end{bmatrix}$$

s.t. eigenvalues of $-EL$: $0, 0, -1, -0.1376, -0.0526$

note: $\ker(-EL) = \ker(L) = \mathcal{S}(\xi)$

Example

$$\dot{x} = -ELx$$

$$\begin{aligned} x(t) &= \mathrm{e}^{-ELt}x(0) \\ &= \mathrm{e}^{VJV^{-1}t}x(0) \\ &= V\mathrm{e}^{Jt}V^{-1}x(0) \end{aligned}$$

$$= [v_1 \ v_2 \ v_3 \ v_4 \ v_5] \begin{bmatrix} \mathrm{e}^{0t} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{0t} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-t} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-0.1376t} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-0.0526t} \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0)$$

$$\to [v_1 \ v_2 \ v_3 \ v_4 \ v_5] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0), \ \text{as } t \to \infty$$

Example

$$\dot{x} = -ELx$$

$$\begin{aligned} x(t) &= \mathrm{e}^{-ELt} x(0) \\ &= \mathrm{e}^{VJV^{-1}t} x(0) \\ &= V\mathrm{e}^{Jt} V^{-1} x(0) \end{aligned}$$

$$= [v_1 \ v_2 \ v_3 \ v_4 \ v_5] \begin{bmatrix} \mathrm{e}^{0t} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{0t} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-t} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-0.1376t} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-0.0526t} \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \\ w_5^\top \end{bmatrix} x(0)$$

$$\to v_1 w_1^\top x(0) + v_2 w_2^\top x(0) \qquad (v_1 = \mathbf{1}, v_2 = \xi)$$

$$= (w_1^\top x(0))\mathbf{1} + (w_2^\top x(0))\xi$$

Example

$$\dot{x} = -ELx$$

if nonzero eigenvalues of $-EL$ are stable

then $x(t) \to \ker(-EL)$ as $t \to \infty$

Theorem

a system of n interacting agents is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{C}$$

target configuration $\xi = [\xi_1 \cdots \xi_n]^\top$

if ξ is generic and $\mathcal{G}$ contains a spanning 2-tree

then there exists E s.t. $\dot{x} = -ELx$

solves 2D formation problem

$$(\forall x(0) \in \mathbb{C}^n)(\exists \xi' \in \mathcal{S}(\xi)) \lim_{t \to \infty} x(t) = \xi'$$

Theorem

Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 2-tree, find a diagonal matrix E s.t. $\dot{x} = -ELx$ solves 2D formation

(i) $\text{rank}(L) = n - 2$ and $\ker(L) = \mathcal{S}(\xi)$

hint: $L\mathbf{1} = L\xi = 0 \Rightarrow \text{rank}(L) \leq n - 2$

spanning 2-tree $\Rightarrow \text{rank}(L) \geq n - 2$

Theorem

Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 2-tree, find a diagonal matrix E s.t. $\dot{x} = -ELx$ solves 2D formation

(ii) $-EL$ has two zero eigenvalues and $\ker(-EL) = \ker(L) = \mathcal{S}(\xi)$

hint: $\text{rank}(E) = n \Rightarrow \text{rank}(EL) = \text{rank}(L)$

Theorem

Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 2-tree, find a diagonal matrix E s.t. $\dot{x} = -ELx$ solves 2D formation

(iii) $n-2$ nonzero eigenvalues of $-EL$ have negative real parts, i.e. stable

hint: $\mathcal{G}$ contains a spanning 2-tree and ξ generic $\Rightarrow$

E exists s.t. $n-2$ nonzero eigenvalues of $-EL$ are stable

[Friedland, 1975]

Theorem

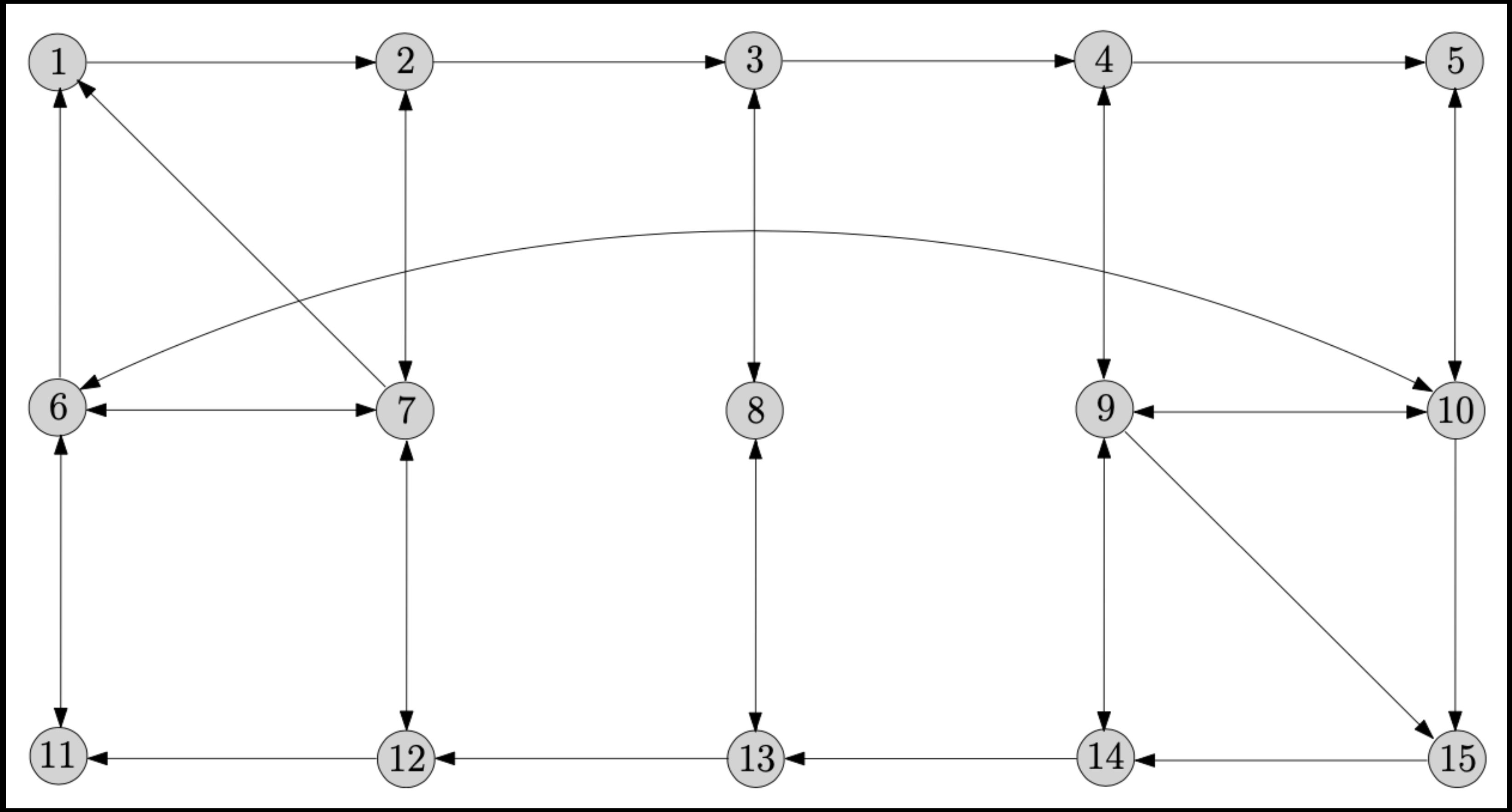
Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 2-tree, find a diagonal matrix E s.t. $\dot{x} = -ELx$ solves 2D formation

(iv) $x(t) \to \ker(-EL) = \mathcal{S}(\xi)$

Example

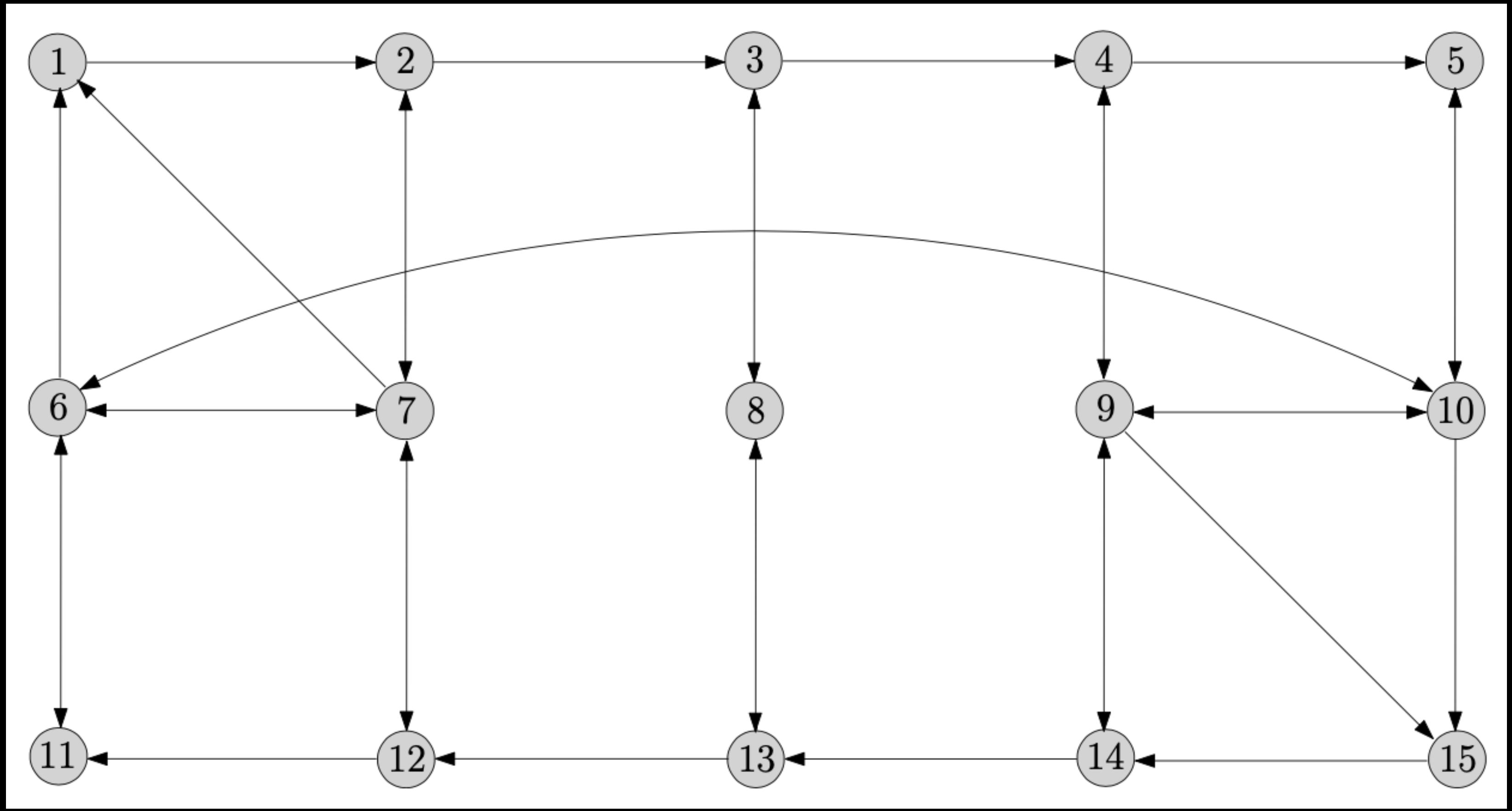
15 networked agents



digraph $\mathcal{G}$ contains a spanning 2-tree

Example

15 networked agents



generic configuration:

$$\xi = [1\ \mathrm{e}^{\frac{2\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{4\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{6\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{8\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{10\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{12\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{14\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{16\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{18\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{20\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{22\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{24\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{26\pi}{15}\mathrm{j}}\ \mathrm{e}^{\frac{28\pi}{15}\mathrm{j}}\ ]^\top$$

Example

15 networked agents