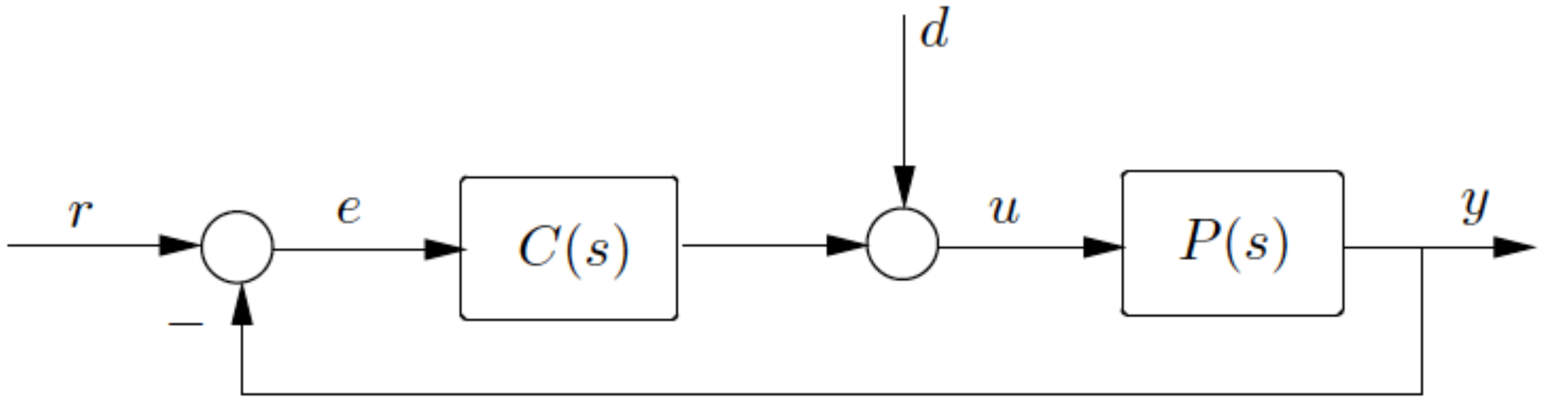
Tracking reference signals

Last week: stability of feedback loop 1

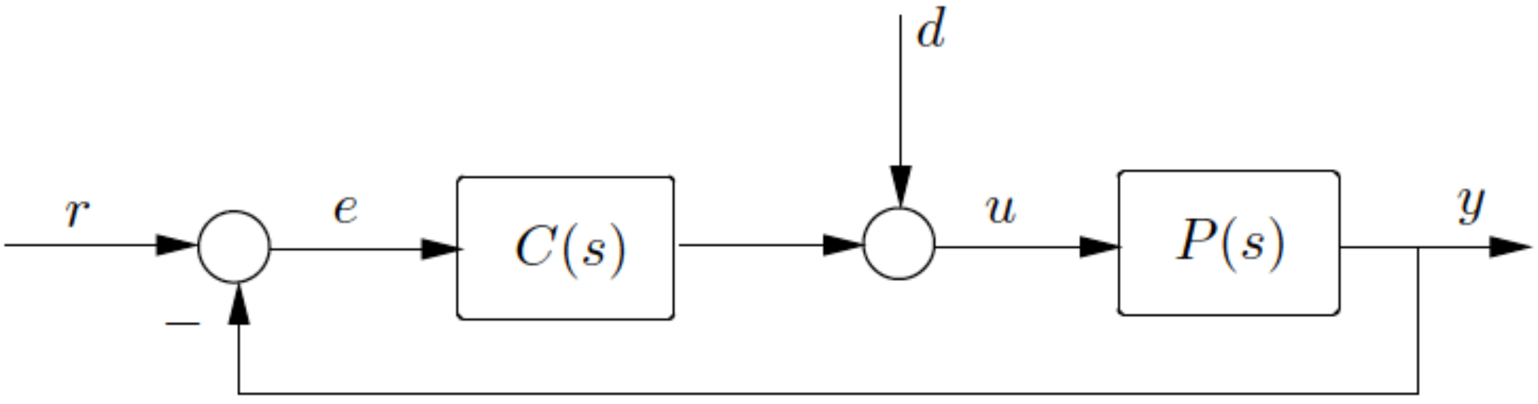


$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_p & B_pC_c \\ -B_cC_p & A_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \begin{bmatrix} 0 & B_p \\ B_c & 0 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} -C_p & 0 \\ 0 & C_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \begin{bmatrix} r \\ d \end{bmatrix}$$

Feedback loop (system) is stable if
the eigenvalues of A_{cl} all have negative real parts

Last week: stability of feedback loop 2

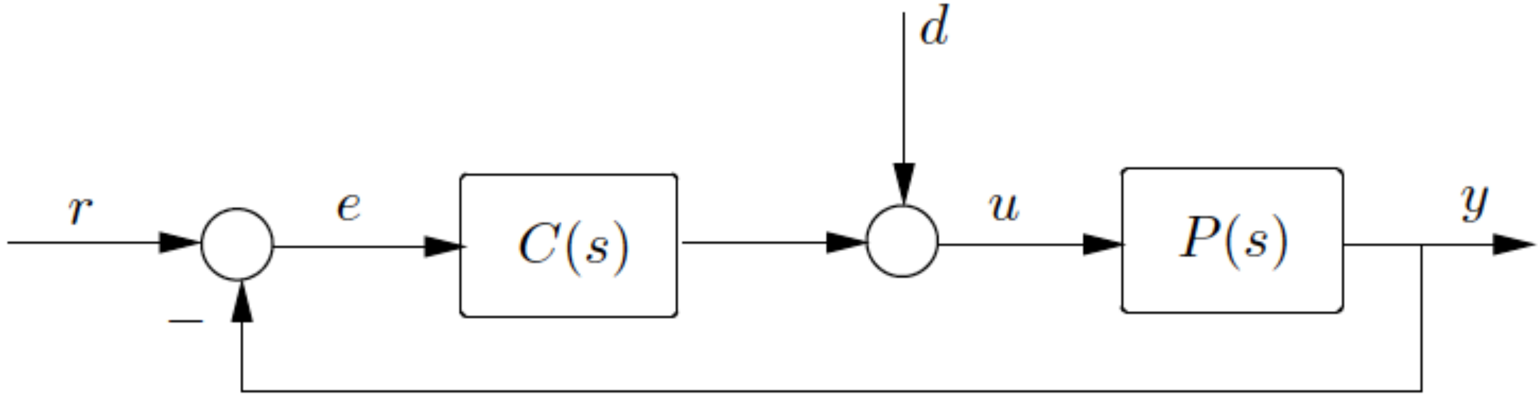


$$P = \frac{N_p}{D_p},\ C = \frac{N_c}{D_c}$$

$$\begin{bmatrix} E \\ U \end{bmatrix} = \frac{1}{D_p D_c + N_p N_c} \begin{bmatrix} D_p D_c & -D_c N_p \\ D_p N_c & D_p D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

The closed-loop system is stable if and only if
the zeros of $D_p D_c + N_p N_c$ all have negative real parts

Last week: transfer functions of feedback loop

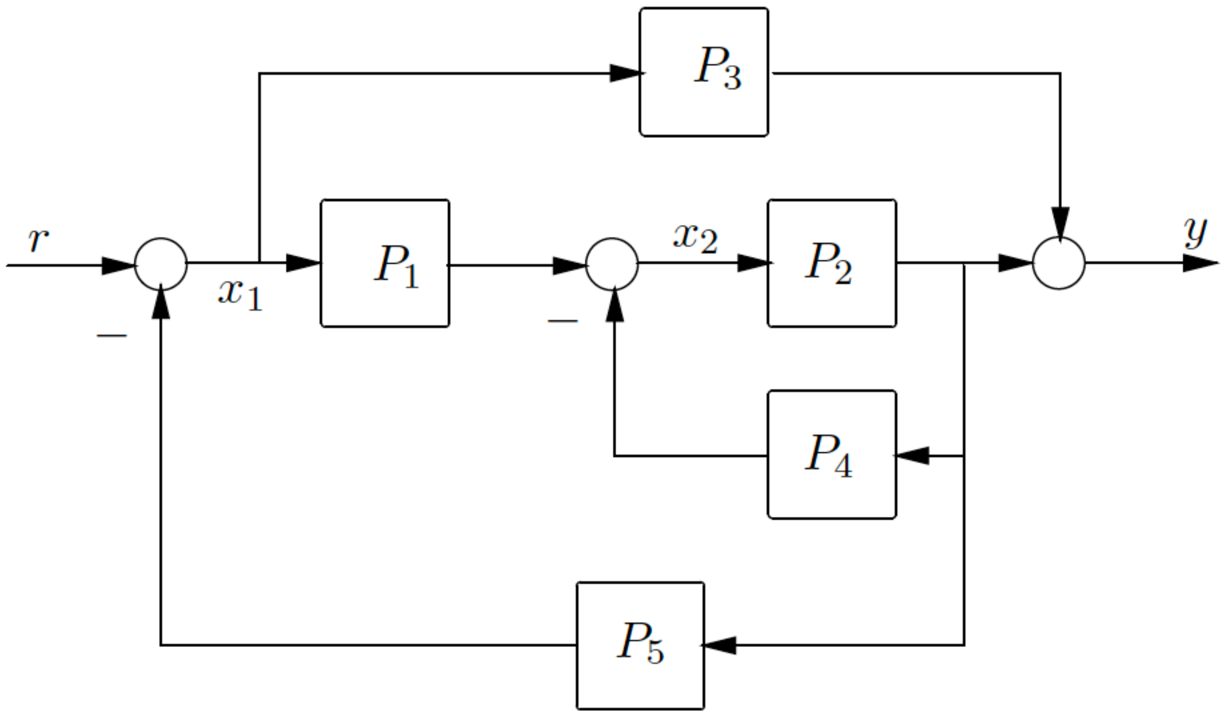


$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}$$

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1+PC} & -\frac{P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

Example



Find transfer function from r to y

At the three summing junctions:

$X_1 = R - P_2 P_5 X_2 \qquad X_2 = P_1 X_1 - P_2 P_4 X_2 \qquad Y = P_3 X_1 + P_2 X_2$

$$\begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$$

Example

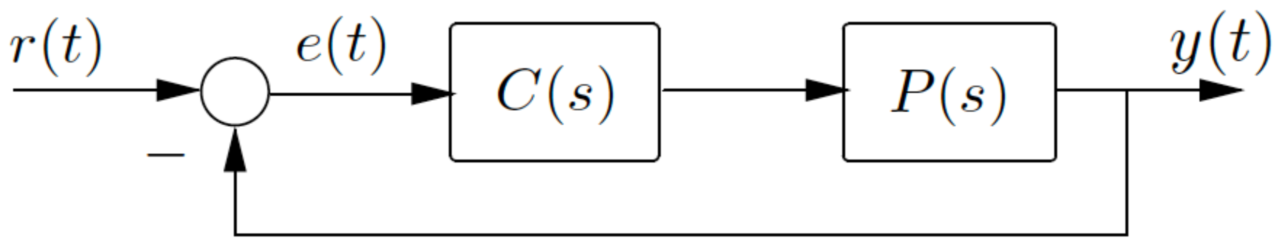
$$\begin{bmatrix} 1 & P_2P_5 & 0 \\ -P_1 & 1+P_2P_4 & 0 \\ -P_3 & -P_2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$$

Solve for Y by Cramer's rule:

$$Y =$$

$$Y = \frac{P_1P_2 + P_3(1+P_2P_4)}{1+P_2P_4+P_1P_2P_5} R$$

Tracking reference signals: setup



Requirement: output $y(t)$ follows a specified reference signal $r(t)$

Hidden requirement: closed-loop stability (always required)

Example

Cruise control of a car: you set a reference speed, say 100km/h and a controller regulates the speed to the setpoint

$P(s) = \frac{1}{s+1}$, $R(s) = \frac{100}{s}$, first trial controller $C(s) = 1$

Closed-loop characteristic polynomial is

Thus the closed-loop system is stable

Transfer function from r to e is $\frac{E(s)}{R(s)} = \frac{1}{1+P(s)C(s)} =$

Thus $E(s) =$

By final-value theorem: $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) =$

Example

Cruise control of a car: you set a reference speed, say 100km/h and a controller regulates the speed to the setpoint

$P(s) = \frac{1}{s+1}$, $R(s) = \frac{100}{s}$, second trial controller $C(s) = 50$

Closed-loop characteristic polynomial is

Thus the closed-loop system is stable

Transfer function from r to e is $\frac{E(s)}{R(s)} = \frac{1}{1+P(s)C(s)} =$

Thus $E(s) =$

By final-value theorem: $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) =$

Example

High controller gain reduces tracking error,
but is expensive over a wide bandwidth

$P(s) = \frac{1}{s+1}$, $R(s) = \frac{100}{s}$, third trial controller $C(s) = \frac{1}{s}$

Closed-loop characteristic polynomial is

Thus the closed-loop system is stable

Transfer function from r to e is $\frac{E(s)}{R(s)} = \frac{1}{1+P(s)C(s)} =$

Thus $E(s) =$

By final-value theorem: $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) =$

Example

Cruise control of a car: you set a reference speed, say 100km/h and a controller regulates the speed to the setpoint

$P(s) = \frac{1}{s+1}$, $R(s) = \frac{100}{s}$, third trial controller $C(s) = \frac{1}{s}$

Note: $R(s)$ has a pole at $s = 0$,
so $R(s)$ is generated by an integrator

The controller is an integrator too

This is the internal model principle:
the controller provides an internal model of the reference

Example

Tracking a ramp signal

$P(s) = \frac{2s+1}{s(s+1)}$, $R(s) = \frac{r_0}{s^2}$, controller $C(s) = \frac{1}{s}$

Closed-loop characteristic polynomial is $s^2(s+1) + 2s + 1$

Thus the closed-loop system is stable

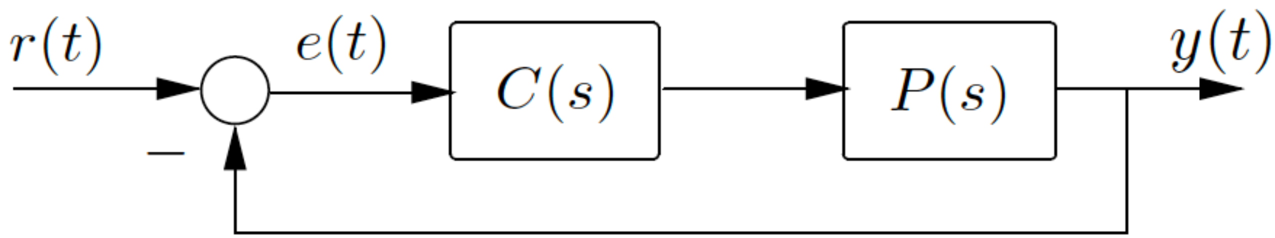
Transfer function from r to e is $\frac{E(s)}{R(s)} = \frac{1}{1+P(s)C(s)} = \frac{s^2(s+1)}{s^2(s+1)+2s+1}$

Thus $E(s) = \frac{s+1}{s^3+s^2+2s+1} r_0$

$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(s+1)}{s^3+s^2+2s+1} r_0 =$

Plant and controller together provide an internal model

Internal model principle



Assume that $P(s)$ is strictly proper, $C(s)$ is proper, and the closed loop is stable.

If $P(s)C(s)$ contains an internal model of the unstable part of $R(s)$, then perfect asymptotic tracking occurs, i.e. $\lim_{t\to\infty} e(t) = 0$

Robustness: as long as the closed loop is stable, perfect tracking occurs under sufficiently small perturbation of $P(s)$

Example

Tracking a sinusoidal signal

$P(s) = \frac{1}{s+1}$, $R(s) = \frac{r_0}{s^2+1}$, controller $C(s) = \frac{s}{s^2+1}$

Closed-loop characteristic polynomial is $(s^2+1)(s+1)+s$

Thus the closed-loop system is stable

Transfer function from r to e is $\frac{E(s)}{R(s)} = \frac{1}{1+P(s)C(s)} = \frac{(s^2+1)(s+1)}{(s^2+1)(s+1)+s}$

Thus $E(s) = \frac{s+1}{(s^2+1)(s+1)+s} r_0$

$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(s+1)}{(s^2+1)(s+1)+s} r_0 =$