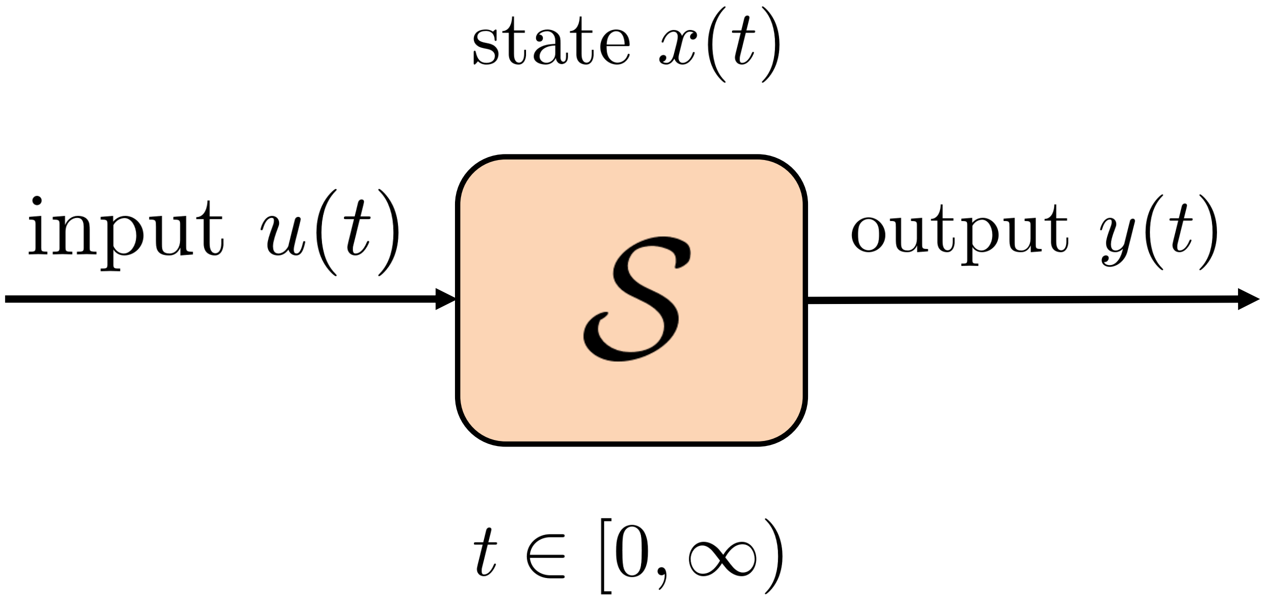
Introduction to State Models

Mathematical models of physical systems

- Kepler's laws of planetary motion
- Newton's laws of motion, law of gravity
- Kirchhoff's laws of circuits
- Maxwell's laws of electromagnetism
- Navier-Stokes equations of fluid flow

Dynamic system and internal state

- Dynamic system: internal state changes over time
- State model: captures such changes

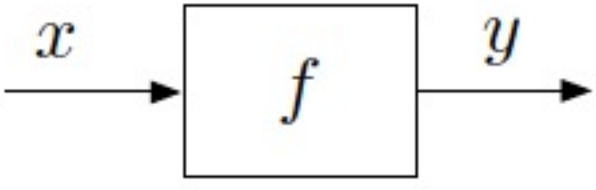


Block Diagrams

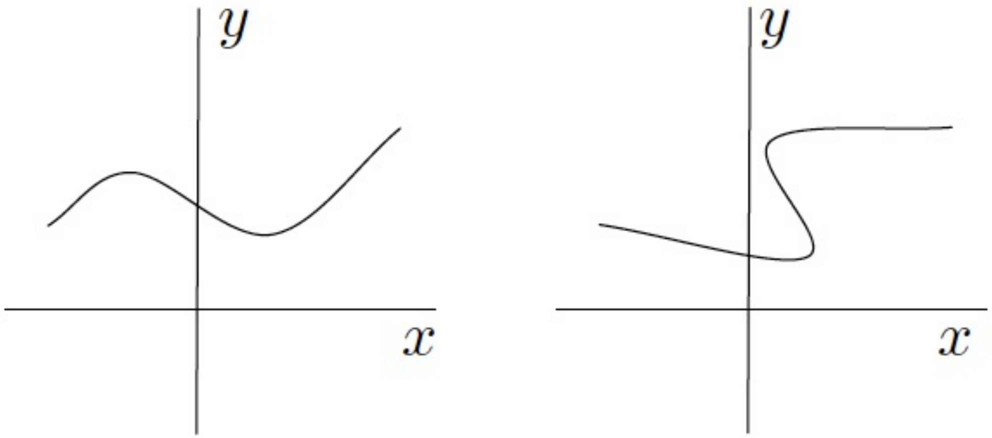
Block diagrams

Consider a function $f : X \to Y$

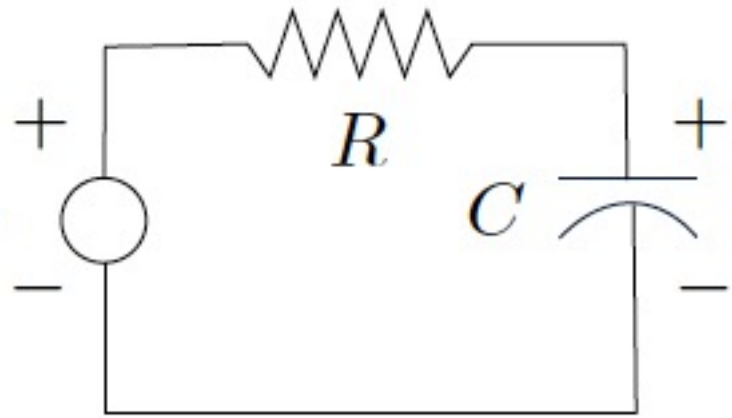
Its block diagram is:



The function $f : \mathbb{R} \to \mathbb{R}$ has a graph in the (x, y) plane



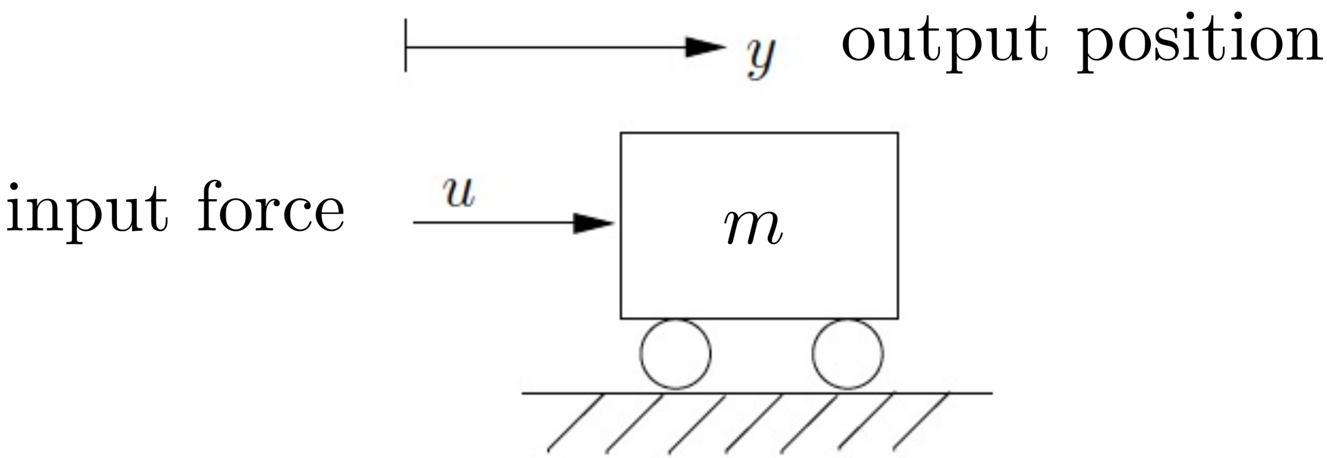
Example: electric circuit

input voltage u  y output voltage

Kirchhoff circuit equation: $RC\dot{y} + y = u$

Its block diagram is:

Example: mechanics

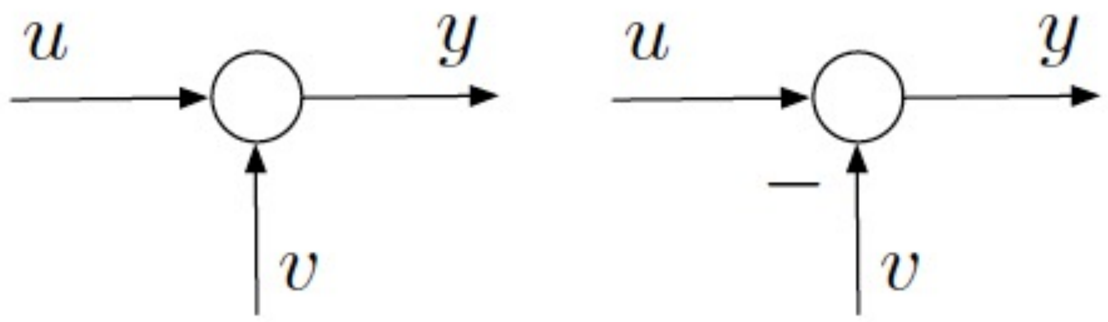


Newton's second law: $u = m\ddot{y}$

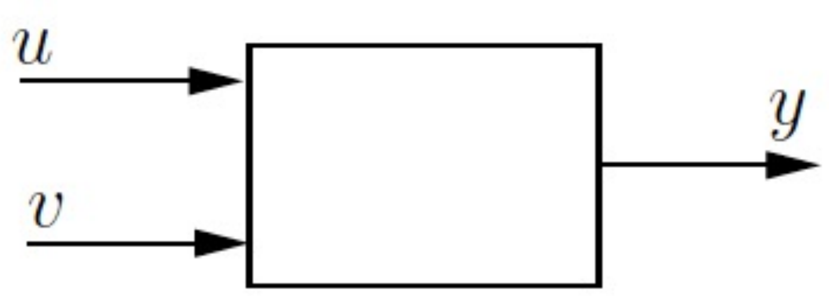
Its block diagram is:

Block diagrams

Summing junctions:

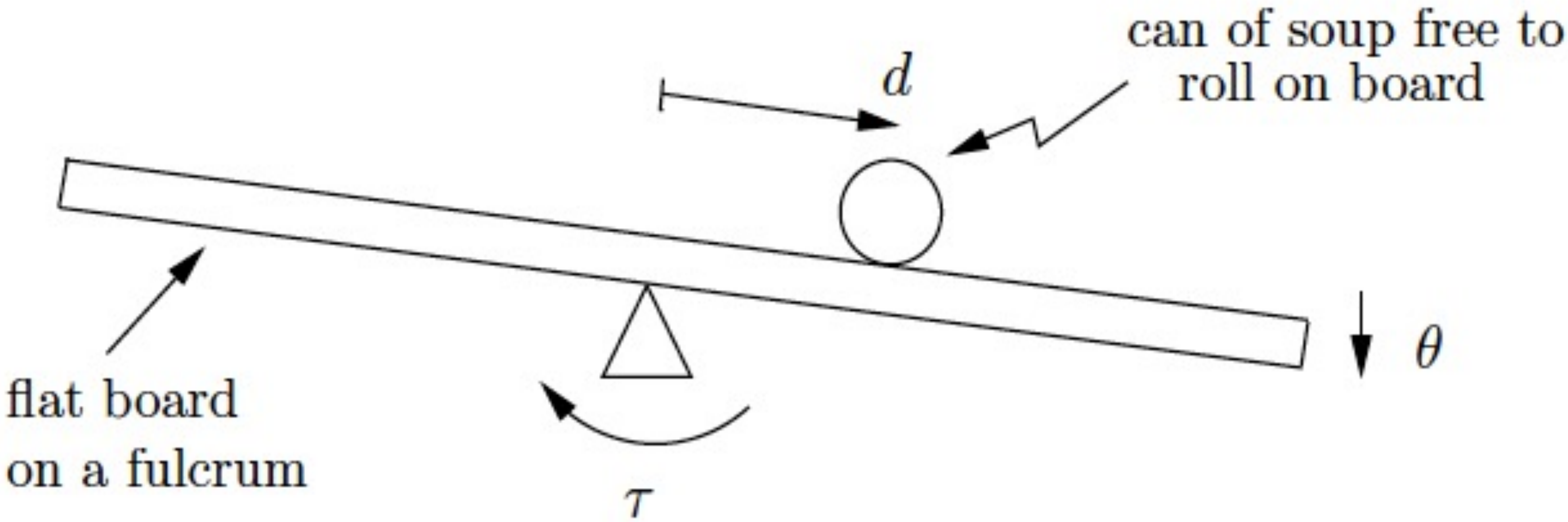


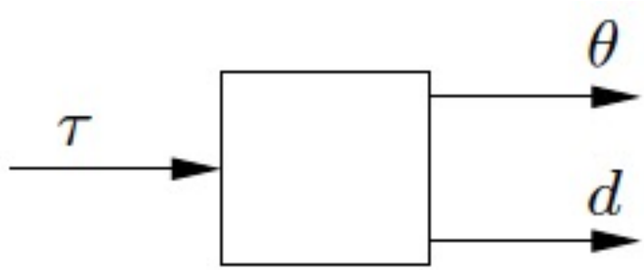
More than one input:



Block diagrams

More than one output:





Recap

A block diagram consists of arrows, boxes, and summing junctions

Arrows represent signals

Arrows without sources represent independent signals

Other arrows represent dependent signals

Boxes represent systems or system components (mathematically functions mapping signals to signals)

Linearity

Linearity

Consider a function $f : X \to Y$, where X, Y are vector spaces over the field of real numbers

In your linear algebra course, f is called a linear function (or linear transformation) if for all vectors x_1, x_2 and all real nubmers a_1, a_2:
$f(a_1 x_1 + a_2 x_2) = a_1 f(x_1) + a_2 f(x_2)$

Linearity

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$

f is linear if and only if $f(x) = Ax$, A is a $m \times n$ matrix

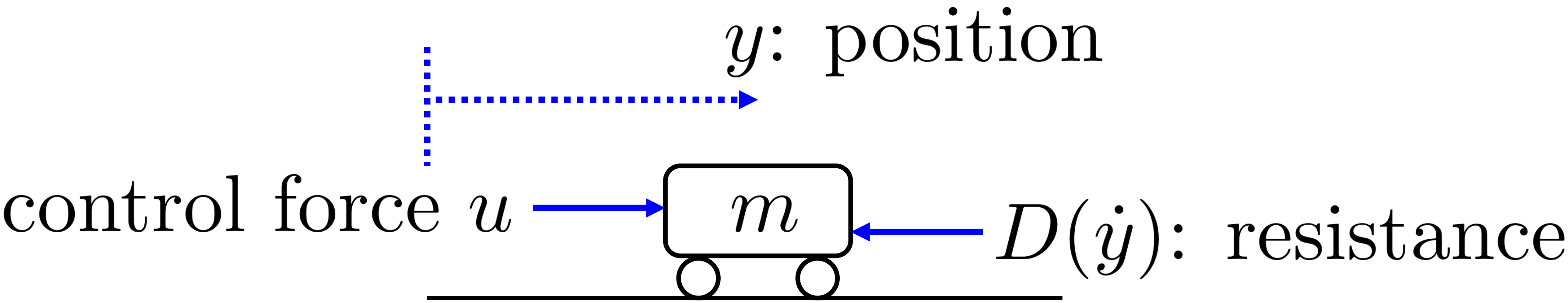
Consider a function $f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m$

f is linear if and only if $f(x, y) = Ax + By$,
A is a $m \times n$ matrix and B is a $m \times l$ matrix

State Models

Example

- Consider a self-driving car:



By Newton's second law: $u - D(\dot{y}) = m\ddot{y}$

This is a constant-coefficient second-order ODE.

Example

ODE: $u - D(\dot{y}) = m\ddot{y}$

Choose state variables: $x_1 = y$, $x_2 = \dot{y}$

Obtain derivatives of state variables:

$$\dot{x}_1 = \dot{y} = x_2$$
$$\dot{x}_2 = \ddot{y} = \frac{1}{m}(u - D(\dot{y}))$$
$$y = x_1$$

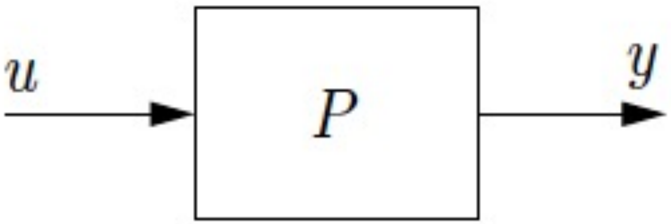
Combine these equations:

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, f(x, u) = \begin{bmatrix} x_2 \\ \frac{1}{m}u - \frac{1}{m}D(x_2) \end{bmatrix}, h(x) = x_1$$

State model

$$\dot{x} = f(x, u)$$
$$y = h(x)$$



constitute a state model of the system (nonlinear, time-invariant)

where $x \in \mathbb{R}^n$: state vector

$u \in \mathbb{R}^m$: control input vector

$y \in \mathbb{R}^p$: measurement output vector

$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

$h : \mathbb{R}^n \to \mathbb{R}^p$

Example

If $D(x_2) = D_0 x_2,\ D_0$ is a constant

then f is linear:

$$f(x,u) = Ax + Bu, A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{D_0}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Since h is linear:

$$h(x) = Cx, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

State model

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $x \in \mathbb{R}^n$: state vector

$u \in \mathbb{R}^m$: control input vector

$y \in \mathbb{R}^p$: measurement output vector

$A \in \mathbb{R}^{n \times n}$: state matrix

$B \in \mathbb{R}^{n \times m}$: input matrix

$C \in \mathbb{R}^{p \times n}$: output matrix

(linear, time-invariant (LTI))

State model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$D \in \mathbb{R}^{p \times m}$: input-output matrix

Block diagram:

Meaning of "state"

The state x at time t should encapsulate all the system dynamics up to t: that is, no additional prior information is required.

For any time t_0 and t_1 ($t_0 < t_1$), knwoing $x(t_0)$ and the input $\{u(t) \mid t_0 \leq t \leq t_1\}$, we can compute $x(t_1)$

Meaning of “state”

For circuits: inductor currents, capacitor voltages

For mechanics: positions, velocities of all masses

State model

Not all dynamic systems have state models:

Differentiator: $y = \dot{u}$

Time delay: $y(t) = u(t - \tau)$

PDE models: e.g. vibrating violin string with input the bow force

Recap

Many dynamic systems have state models:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

This model is nonlinear if either f or h is nonlinear

It is time-invariant, because f, h do not depend on t

The linear time-invariant state model is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$