Laplace transform (cont’d)

Inverse Laplace transform

Given $F(s)$, find $f(t)$.

Ex. $F(s) = \frac{3s+17}{s^2-4}$

Partial fraction expansion:
$F(s) = \frac{c_1}{s-2} + \frac{c_2}{s+2}$, $c_1 =?$, $c_2 =?$

Look up the table:
$f(t) = \frac{23}{4}\mathrm{e}^{2t} - \frac{11}{4}\mathrm{e}^{-2t}$

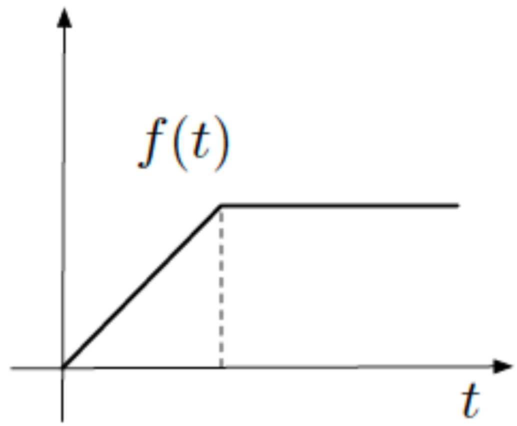
Note: we know nothing about the value of $f(t)$ for $t < 0$

Linearity

Mapping $f(t) \to F(s)$ is linear

$$(\forall a_1, a_2 \in \mathbb{R}) a_1 f_1(t) + a_2 f_2(t) \to a_1 F_1(s) + a_2 F_2(s)$$

Linearity



$$f(t) = f_1(t) + f_2(t)$$

$$f_1(t) = t$$

$$f_2(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ -f_1(t-1), & t > 1 \end{cases}$$

$$F_1(s) =$$

Linearity

$$f_2(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ -f_1(t-1), & t > 1 \end{cases}$$

$$F_2(s) =$$

$$F(s) = F_1(s) + F_2(s)$$

Superposition

Laplace transform of a sum equals the sum of Laplace transforms

If $f(t) = g(t) + h(t)$, then $F(s) = G(s) + H(s)$

Convolution

Assume $g(t)$, $h(t)$ equal zero for $t < 0$.

The *convolution* of $g(t)$ and $h(t)$ is

$$f(t) = \int_{-\infty}^{\infty} g(t-\tau)h(\tau)d\tau$$

Notation $f(t) = g(t) * h(t)$

Note: $f(t) = 0$ for $t < 0$

If $f(t) = g(t) * h(t)$, then $F(s) = G(s)H(s)$

Convolution

Proof: All the integrals range from $-\infty$ to ∞.

$$
\begin{aligned}
F(s) &= \int f(t)\mathrm{e}^{-st}dt \\
&= \int \int g(t-\tau)h(\tau)d\tau\,\mathrm{e}^{-st}dt \\
&= \int \int g(t-\tau)h(\tau)\mathrm{e}^{-st}dt\,d\tau \\
(r := t-\tau) \qquad &= \int \int g(r)h(\tau)\mathrm{e}^{-s(\tau+r)}dr\,d\tau \\
&= \int \int g(r)\mathrm{e}^{-sr}dr\,h(\tau)\mathrm{e}^{-s\tau}d\tau \\
&= G(s)\int h(\tau)\mathrm{e}^{-s\tau}d\tau \\
&= G(s)H(s)
\end{aligned}
$$