Stability of Feedback Loop

Last week: condition for stability

The system modeled by

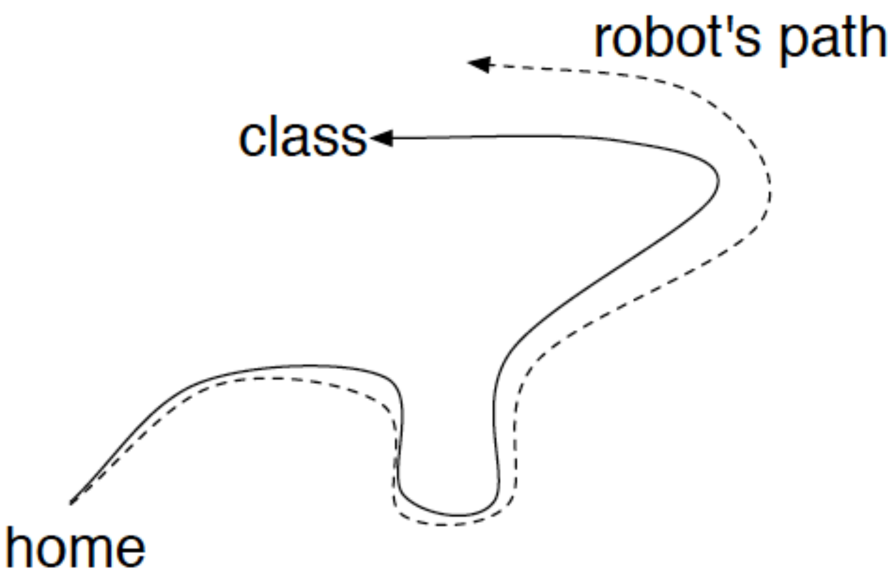
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is stable if
the eigenvalues of A all have negative real parts
(i.e. zeros of $\det(sI - A)$ all have negative real parts)

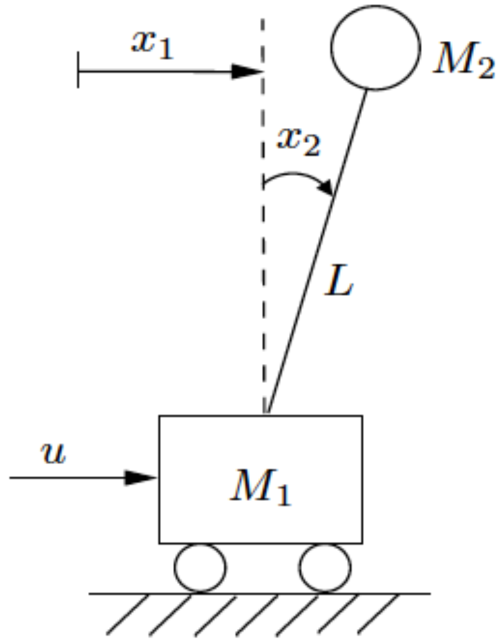
Intuition

Control systems are most often based on feedback

Imagine programming a robot to go from home to class without feedback (no vision/GPS sensors)



Example



pendulum

control input

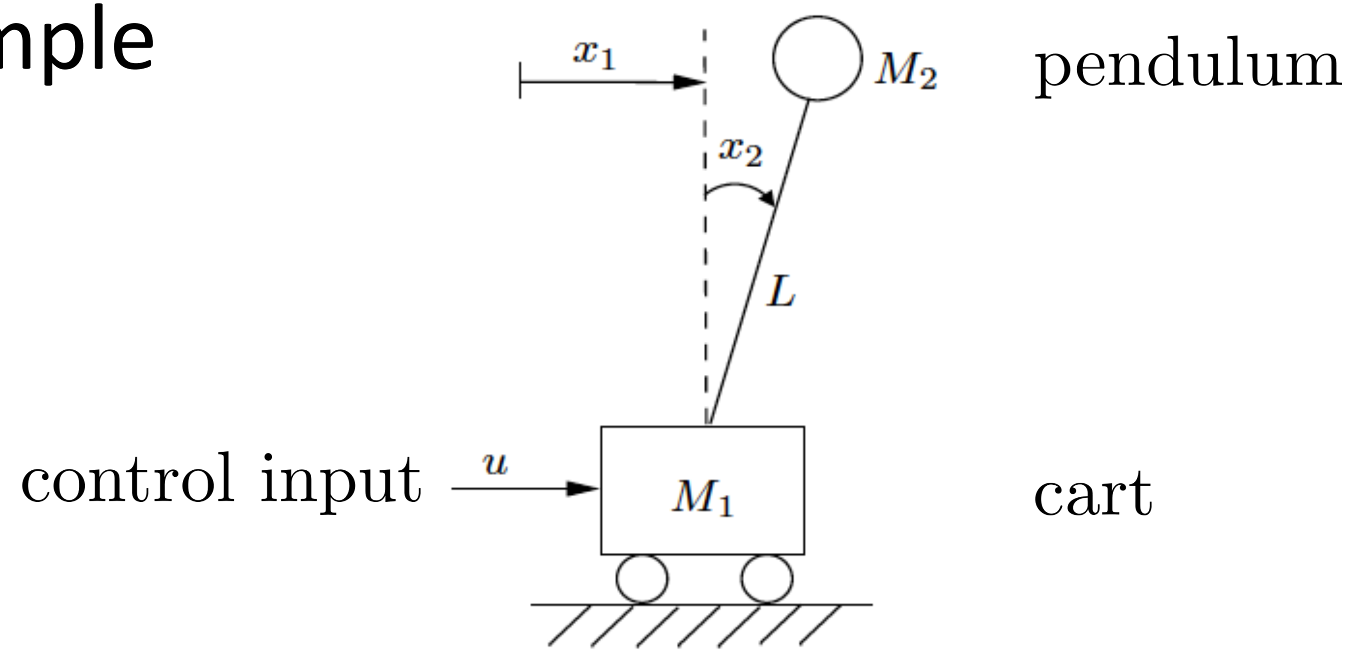
cart

$$M_1 = 1,\ M_2 = 2,\ L = 1,\ g = 9.8$$

Taking $x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$, we get the linearlized state model

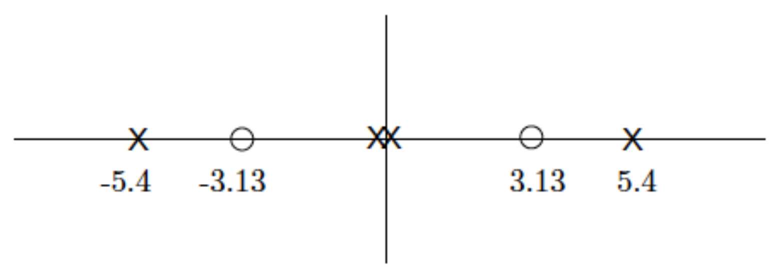
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -19.6 & 0 & 0\\ 0 & 29.4 & 0 & 0\end{bmatrix}}_{A_p} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ 1\\ -1\end{bmatrix}}_{B_p} u,\ \ y = \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}}_{C_p} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

Example



Transfer function (from u to y):

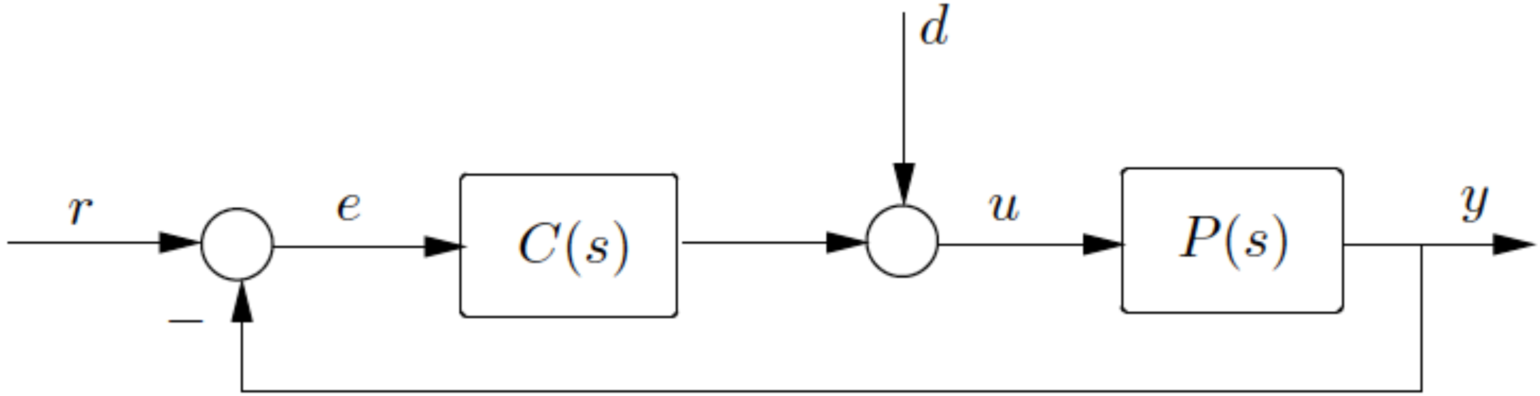
$$P(s) = C_p(sI - A_p)^{-1}B_p$$
$$= \frac{1}{\det(sI - A_p)} C_p \mathrm{adj}(sI - A_p) B_p$$
$$= \frac{s^2 - 9.8}{s^2(s^2 - 29.4)}$$



this zero makes control difficult

poles: $0, 0, 5.4, -5.4$; zeros: $3.1, -3.1$

Standard feedback loop



$P(s)$: plant transfer function

$C(s)$: controller transfer function

$r(t)$: reference (or command) input

$e(t)$: error

$d(t)$: disturbance

$u(t)$: plant input

$y(t)$: plant output

Example

A solution controller (obtained by an advanced method):

$$C(s) = \frac{10395s^3+54126s^2-13375s-6687}{s^4+32s^3+477s^2-5870s-22170}$$

A state realization of $C(s) = C_c(sI - A_c)^{-1}B_c$:

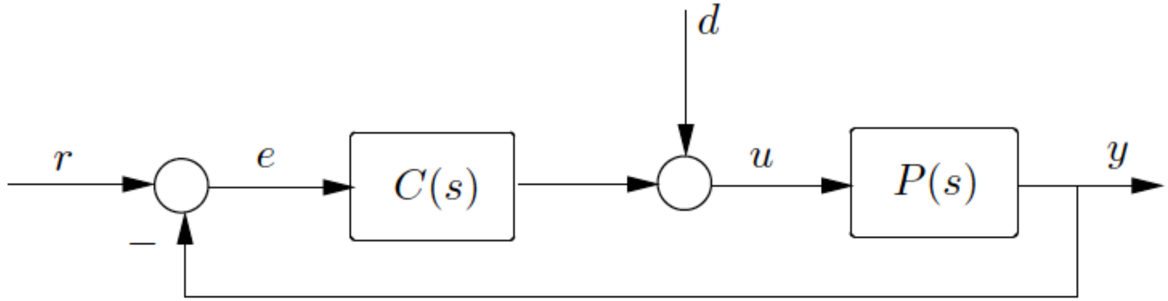
$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 22170 & 5870 & -477 & -32 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

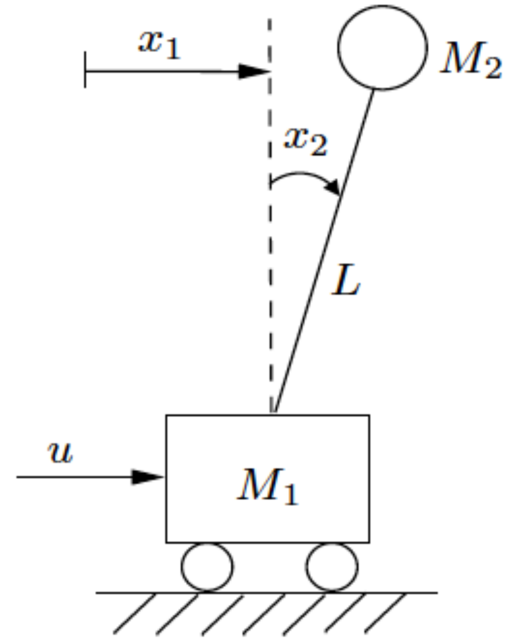
$$C_c = \begin{bmatrix} -6687 & -13375 & 54126 & 10395 \end{bmatrix}$$

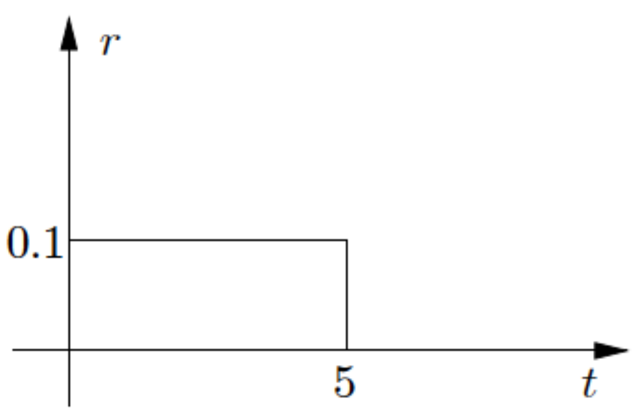
This controller $C(s)$ itself is unstable (check!)

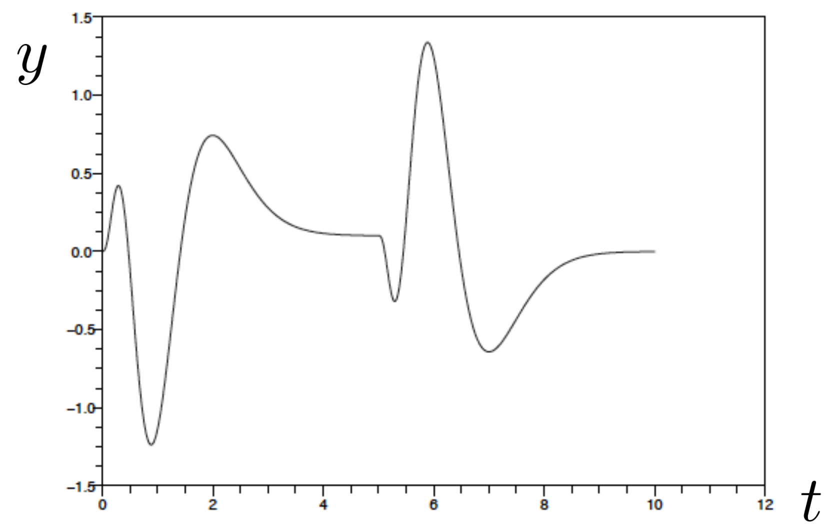
Standard feedback loop of the controller and plant is stable (to be defined)

Example

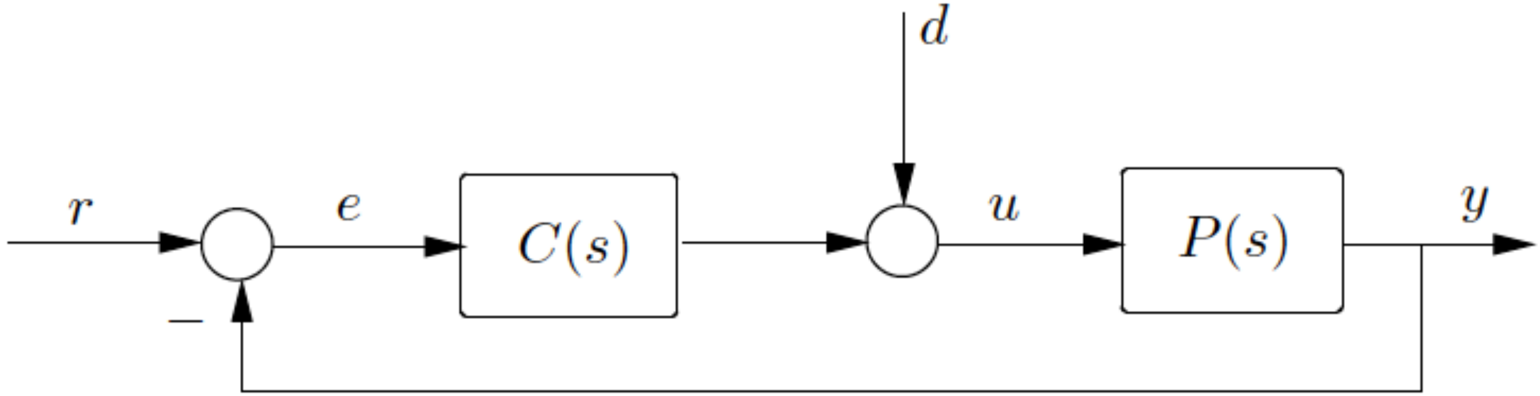








Standard feedback loop



View the system having inputs (r, d) and outputs (e, u)

Let the states of the plant and controller be x_p, x_c

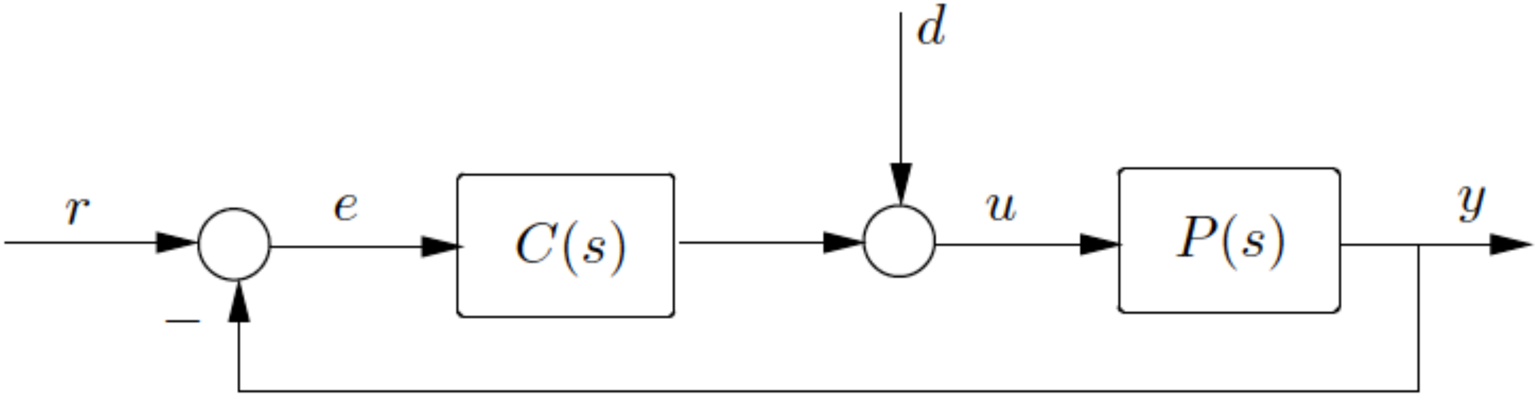
Take the state of the feedback loop to be $x_{cl} = (x_p, x_c)$

$$\dot{x}_p = A_p x_p + B_p u,\ y = C_p x_p$$

$$\dot{x}_c = A_c x_c + B_c e,\ u - d = C_c x_c$$

$$e = r - y$$

Feedback stability



$$\begin{bmatrix}\dot{x}_p\\ \dot{x}_c\end{bmatrix} = \begin{bmatrix}A_p & B_pC_c\\ -B_cC_p & A_c\end{bmatrix}\begin{bmatrix}x_p\\ x_c\end{bmatrix} + \begin{bmatrix}0 & B_p\\ B_c & 0\end{bmatrix}\begin{bmatrix}r\\ d\end{bmatrix}$$

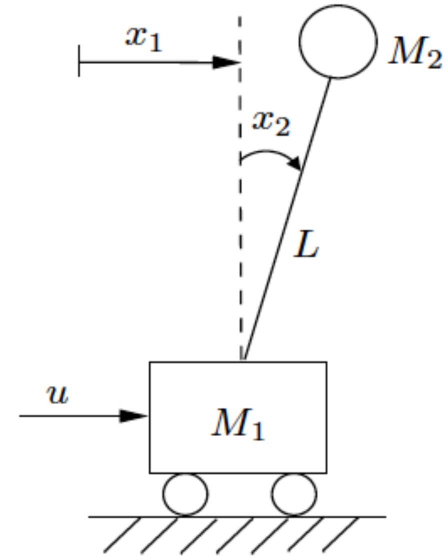
$$\begin{bmatrix}e\\ u\end{bmatrix} = \begin{bmatrix}-C_p & 0\\ 0 & C_c\end{bmatrix}\begin{bmatrix}x_p\\ x_c\end{bmatrix} + \begin{bmatrix}r\\ d\end{bmatrix}$$

Feedback loop (system) is stable if
the eigenvalues of A_{cl} all have negative real parts

Example

$$A_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

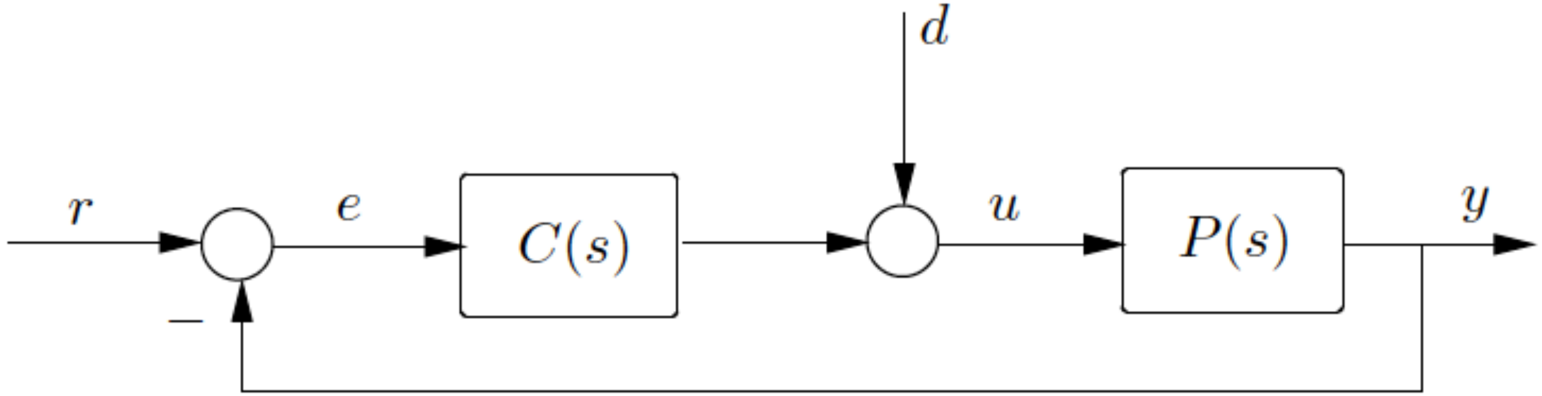


$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 22170 & 5870 & -477 & -32 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_c = \begin{bmatrix} -6687 & -13375 & 54126 & 10395 \end{bmatrix}$$

$$A_{cl} = \begin{bmatrix} A_p & B_p C_c \\ -B_c C_p & A_c \end{bmatrix}$$

Standard feedback loop



$P(s)$: plant transfer function; rational and strictly proper

$C(s)$: controller transfer function; rational and proper

$r(t)$: reference (or command) input

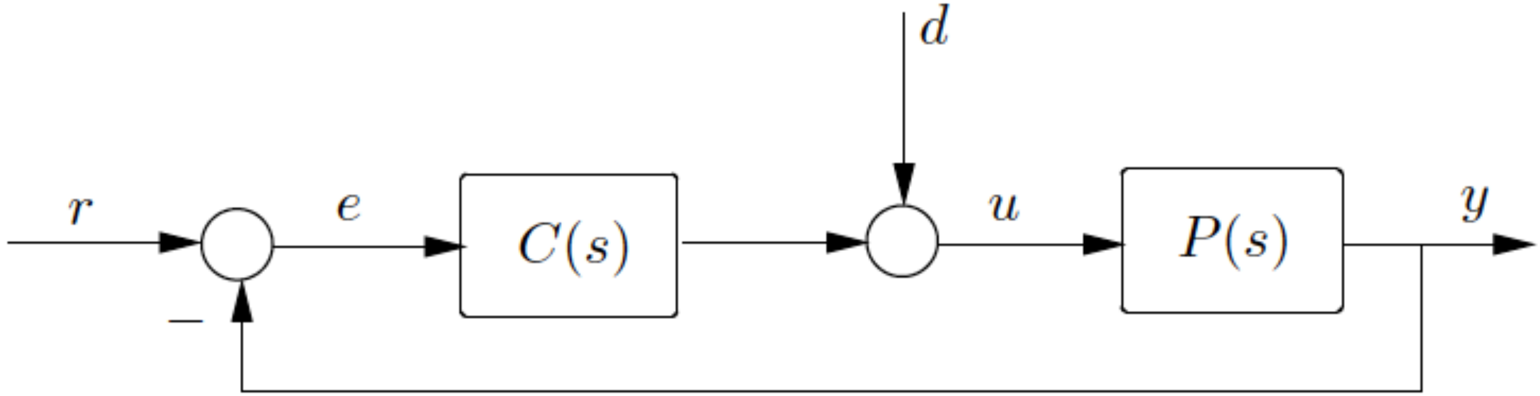
$e(t)$: error

$d(t)$: disturbance

$u(t)$: plant input

$y(t)$: plant output

Standard feedback loop: transfer function



View the system having inputs (r, d) and outputs (e, u)

At the two summing junctions:

$$E(s) = R(s) - P(s)U(s)$$

$$U(s) = D(s) + C(s)E(s)$$

$$\begin{bmatrix} 1 & P(s) \\ -C(s) & 1 \end{bmatrix} \begin{bmatrix} E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$

Standard feedback loop: transfer function

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}$$

$$P = \frac{N_p}{D_p},\ C = \frac{N_c}{D_c}$$

(Note: N_p, D_c or D_p, N_c may have common factors)

$$\begin{bmatrix} 1 & \frac{N_p}{D_p} \\ -\frac{N_c}{D_c} & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}$$

$$\begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} 1 & \frac{N_p}{D_p} \\ -\frac{N_c}{D_c} & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$\begin{bmatrix} D_p & N_p \\ -N_c & D_c \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

Standard feedback loop: transfer function

$$\begin{bmatrix} D_p & N_p \\ -N_c & D_c \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} D_p & N_p \\ -N_c & D_c \end{bmatrix}^{-1} \begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$= \frac{1}{D_p D_c + N_p N_c} \begin{bmatrix} D_c & -N_p \\ N_c & D_p \end{bmatrix} \begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$= \frac{1}{D_p D_c + N_p N_c} \begin{bmatrix} D_p D_c & -D_c N_p \\ D_p N_c & D_p D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

Feedback stability

$$\begin{bmatrix} E \\ U \end{bmatrix} = \frac{1}{D_p D_c + N_p N_c} \begin{bmatrix} D_p D_c & -D_c N_p \\ D_p N_c & D_p D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

Recall:

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = A_{cl} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + B_{cl} \begin{bmatrix} r \\ d \end{bmatrix}$$

$$\begin{bmatrix} e \\ u \end{bmatrix} = C_{cl} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + D_{cl} \begin{bmatrix} r \\ d \end{bmatrix}$$

Feedback stability

$$\begin{bmatrix} E \\ U \end{bmatrix} = \frac{1}{D_p D_c + N_p N_c} \begin{bmatrix} D_p D_c & -D_c N_p \\ D_p N_c & D_p D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

Recall:

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = A_{cl} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + B_{cl} \begin{bmatrix} r \\ d \end{bmatrix}$$

$$\begin{bmatrix} e \\ u \end{bmatrix} = C_{cl} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + D_{cl} \begin{bmatrix} r \\ d \end{bmatrix}$$

$$\begin{bmatrix} E \\ U \end{bmatrix} = \frac{1}{\det(sI - A_{cl})} \left(C_{cl} \mathrm{adj}(sI - A_{cl}) B_{cl} + \det(sI - A_{cl}) D_{cl} \right) \begin{bmatrix} R \\ D \end{bmatrix}$$

Two polynomials $D_p D_c + N_p N_c$ and $\det(sI - A_{cl})$ are equivalent

closed-loop characteristic polynomial

Feedback stability

$$\begin{bmatrix} E \\ U \end{bmatrix} = \frac{1}{D_pD_c + N_pN_c} \begin{bmatrix} D_pD_c & -D_cN_p \\ D_pN_c & D_pD_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

The closed-loop system is stable if and only if
the zeros of $D_pD_c + N_pN_c$ all have negative real parts

Example

$P(s) = \frac{1}{s-1}$, $C(s) = K$

Closed-loop characteristic polynomial:

Thus the closed-loop system is stable if and only if

Example

$P(s) = \frac{1}{s^2-1}$, $C(s) = \frac{s-1}{s+1}$

($P(s)$ has an unstable pole, which is canceled by $C(s)$)

Closed-loop characteristic polynomial:

Thus the closed-loop system is unstable

Canceling unstable poles does not work

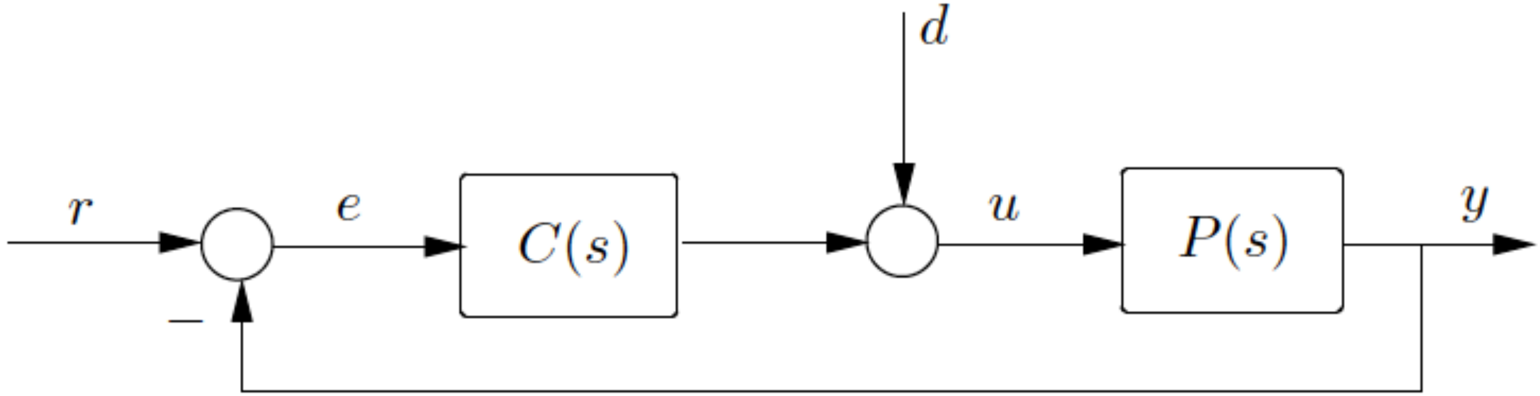
An unstable plant cannot be stablilized by canceling its unstable poles

If $P(s) = \frac{N_p(s)}{D_p(s)}$ has a pole at $s = p$, $\mathrm{Re}(s) \geq 0$, and $C(s) = \frac{N_c(s)}{D_c(s)}$ has a zero at $s = p$

then the closed-loop characteristic polynomial $D_p D_c + N_p N_c$ still has a zero at $s = p$

An unstable plant can be stablilized by moving its unstable poles to the left half-plane

Robustness of feedback stability

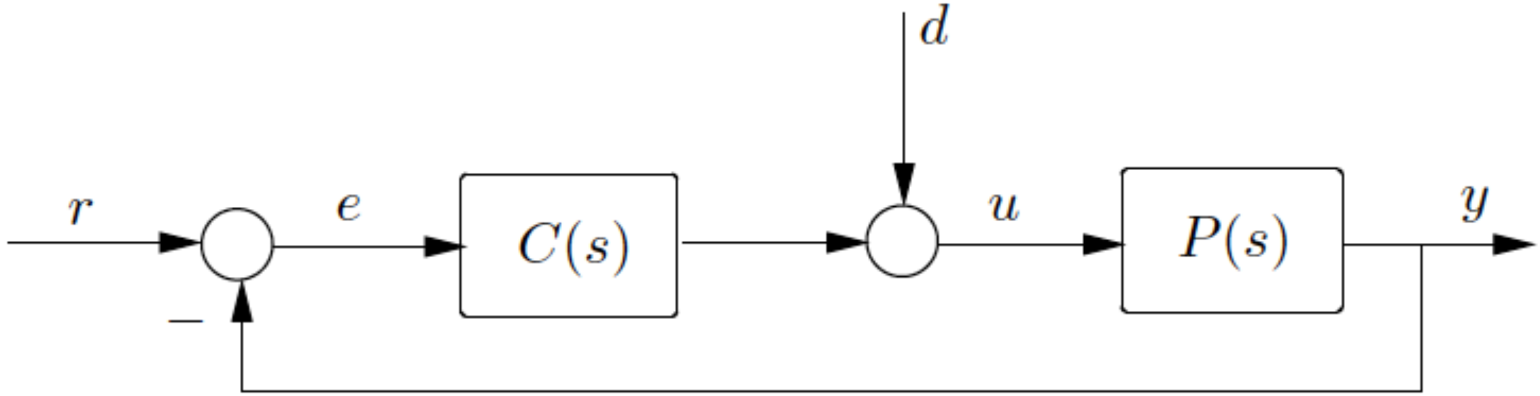


Suppose that the closed-loop system is stable,
i.e. zeros of $D_p D_c + N_p N_c$ all have negative real parts

If we slightly perturb the coefficients of N_p, N_c,
the closed loop will still be stable

This is because zeros of a polynomial are continuous functions
of the coefficients of the polynomial

Standard feedback loop: transfer function



$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}$$

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1+PC} & -\frac{P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$